APPROACHES FOR MACHINE LOADING AND PRODUCT MIX ANALYSIS FOR SINGLE AND MULTI-STAGE PRODUCTION SYSTEMS UTILIZING GT CONCEPT

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CERTIFICATE

This is to certify that this work entitled, 'Approaches for Machine Loading and Product Mix Analysis for Single and Multi-Stage Production Systems Utilizing GT Concept', by Mr. Anil Kumar Agrawal, has been carried out under my supervision and that it has not been submitted elsewhere for the award of a degree.

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CONTENTS

Chapter			Page

ABSTRACT

This thesis considers the problem of determining optimal product mix and machine loading for single and multi-stage production systems. It is assumed that the Group Technology (GT) concepts have been utilized to classify the jobs under consideration for manufacture on the production systems into groups. Each job has a lot size comprising of identical units and each production stage has only one machine. In case of multi-stage production systems, it is further assumed that the job operations do not have any technological ordering. Corresponding to each stage, the job available for processing either have prespecified production times or the production times are to be determined optimally considering machining speeds as decision variables. The optimization is to be performed considering maximization of production rate or maximization of total profit.

For each one of the above stated cases, an optimizing algorithm which uses the concepts of branch and bound procedure is developed.

In order to save on the computational effort involved in the use of branch and bound procedure, for each case a heuristic method has been developed. The heuristic method helps in reducing the size of the problem to be handled by

the branch and bound procedure. Illustrative examples are given to explain the various steps of the optimizing and heuristic algorithms.

The optimizing and heuristic algorithms have been computerized and implemented on DEC-1090 Computer System. Based on the limited experience of solving the numerical examples, it was observed that the heuristic algorithms have computational superiority over the optimizing algorithms.

CHAPTER I

INTRODUCTION

1.1 GROUP TECHNOLOGY:

Group Technology is a very progressive method of organising production and especially it is becoming very popular in those industries which are engaged in medium and small batch productions. Group Technology concept identifies and makes use of sameness or identical nature of operation processes and parts in design and manufacture. Similar parts are grouped together into one part family based on similarity of their design and dimensions, geometrical shapes and technological requirements, and according to the needs to process designated part families, machine group or cells are formed.

1.2 REVIEW OF PREVIOUS WORK:

Analytical methods suggested by Burbridge [1] take into consideration those informations which are contained in route cards. However, he does not consider design and geometrical similarity. Batra and Rajgopalan [2] also use informations given in route card for each job and using graph theoretic approach they form machine cells. McAuley [3] uses similarity coefficient for every pair of machines to form group of machines.

El-Essawy[4] does component flow analysis to form machine and component grouping.

Many researchers have proposed models to schedule products on machines available in machine cells. Petrov [5] has developed four interrelated scheduling models for different type of route sequences and component flows. This approach does not guarantee optimality mathematically. Ham and Hitomi [6] have suggested a technique to schedule various grouped products on multi-stage production system.

In addition to the problem of group formation and group scheduling, machine loading and product mix decisions represent major problem areas for group production planning and scheduling. Machine loading problem determines the total number of units of various jobs to be processed within limited capacity of various production facilities available, while product mix decision determines the kinds of jobs to be processed within limited production time. Hitomi and Ham [7,8] have considered the problem from the view point of Group Technology for a single machine production system. They have presented models for two distinct situations. In the first case, the unit production time is prespecified [7], while in the other case, it is dependent on the machining speed which is treated as a decision variable [8]. They maximize total number of units produced of various products and in process do not account for weightages

or importance of various products to be produced. Moreover, the experience of the author in using Hitomi and Ham's algorithm on a number of problem suggests that their algorithm does not yield optimal solution necessarily as claimed by them.

1.3 SCOPE OF PRESENT STUDY:

Basically, this thesis is concerned with the machine loading and product-mix decision problem considered by Hitomi and Ham [7,8] with added features to make the problem more realistic. In real life, single stage production environment is not very common. Further, besides maximizing the total amount of production, the management may desire to evolve a product mix and machine loading strategy which maximizes the profit. Therefore, in the present work, an attempt is made to develop models and solution methodologies for multi-stage situation with maximization of total production or maximization of profit as the criterion of optimization, considering the unit production time as prespecified or dependent on the machining speed (a decision variable). In general, the solution methodology exploits the branch and bound concepts for its development. Since, the use of branch and bound approach involves considerable effort, heuristic based computationally efficient solution methodologies which initially reduce the size of the problem and then find the solution applying branch and bound concept to the largely reduced problem, are also presented.

Further, the algorithm for single machine loading problem and product mix decision of Hitomi and Ham [7,8] has been modified. The modified algorithm ensures optimal solution to the problem.

1.4 ORGANISATION OF THESIS:

The cases of single stage production system have been dealt in Chapter II. In this chapter besides the modified version of algorithm of Hitomi and Ham [7,8], optimal and heuristic solution procedures for maximization of profit are also presented. Various variations of the multi-stage production system have been considered in Chapter III. Numerical examples are presented to illustrate the solution methodologies proposed for each of the variations of the single and multi-stage production systems considered. In Chapter IV, conclusions based on the present study alongwith suggestions for further work are presented.

CHAPTER II

SINGLE STAGE PROBLEM

2.1 STATEMENT OF THE PROBLEM:

Consider a single stage production system with limited available production time. N types of jobs are available for production. Based on part-family concept of group technology, these jobs are classified into N_g groups. Let groups G_i $(i = 1,2,\ldots, N_g)$ contains N_i jobs.

The objective is to determine an optimal machine loading and product mix such that within the available production time, the quantities produced of each job do not exceed the lot size limit. The optimization is carried out to maximize either the production rate or the total profit.

Corresponding to each group, there is a group production time which is comprised of the group set-up time and the sum of the job production times of the jobs in this group. The job production time includes the job set-up time and the time required to produce the various units of that job. The unit production time required to produce single unit of any job may be prespecified or dependent upon the cutting speed (a decision variable).

2.2 ASSUMPTIONS:

The various assumptions are:

i) The single stage production system contains only one machine.

ii) Only one unit of the job can be processed on the machine at a time.

iii) A limited amount of time is available for production on the machine. This is the actual productive time available on the machine and as such do not include time lost due to breakdowns, etc.

iv) Each unit of the job must be processed to completion.

v) All the units of a job are produced together. Similarly, all the jobs belonging to same group are produced together.

2.3 NOMENCLATURE:

Following notations are used for the development of the mathematical models:

a_{ij} preparation time for job J_{ij} (min/pc)

b_{ij} tool replacement time for job J_{ij} (min/pc)

B_{ij} profit per unit of job J_{ij} (Rs./pc)

C_{ij} 1-minute tool life machining speed for job J_{ij} (m/min)

C_u total capacity of the machine utilized (min.)

d available production time (min)

E_{ij} high-efficiency machining speed range for job J_{ij} (m/min)

G_i i-th group or part family

i group index ($= 1,2,\ldots, N_g$)

j job index ($= 1,2,\ldots, N_i$)

J_{ij} j-th job in i-th group

l_{ij} lot size of J_{ij} (pcs)

n_{ij} slope constant of Taylor tool life equation

N total number of jobs available for production

N_g total number of part-families or groups

N_i total number of jobs in group G_i

p_{ij} unit production time for J_{ij} (min/pc)

$p_{ij}^{(c)}$ minimum-production-cost unit production time for job J_{ij} (min/pc)

$p_{ij}^{(t)}$ maximum-production-rate unit production time for job J_{ij} (min/pc)

$p_{ij}(v_{ij})$ cutting speed dependent unit production time (min/pc)

P_{ij} job production time for J_{ij} (min)

$P_{ij}^{(t)}$ maximum-production-rate job production time for J_{ij} (min)

$q_{ij}(v_{ij})$ speed dependent unit production cost (Rs/pc)

s_{ij} job set-up time for J_{ij} (min)

S_i Group set-up time for group G_i (min)

t_{ij} actual machining time for J_{ij} (min/pc)

T_{ij} tool life for J_{ij} (min/edge)

U set of all parts available for production $\{ J_{ij}/j = 1,2,\ldots, N_i ; i = 1,2,\ldots, N_g\}$

v_{ij} machining speed for J_{ij} (m/min)

$v_{ij}^{(c)}$	minimum-production-cost machining speed for J_{ij} (m/min)
$v_{ij}^{(c)}$	high efficiency machining speed for J_{ij} (m/min)
$v_{ij}^{(t)}$	maximum-production rate machining speed for J_{ij} (m/min)
w_{ij}	quantity of J_{ij} to be produced (pcs)
x_{ij}	0-1 type variable for J_{ij}
X_i	0-1 type variable for group G_i
Y	total production cost (Rs.)
Y'	total variable production cost (Rs.)
Z	value of objection function
$\underline{Z}$	lower bound on objective function value
$\bar{Z}$	upper bound on objective function value
Z_{max}	maximum value of objective function
α	direct labour cost and overhead (Rs./min)
β_{ij}	machining overhead for J_{ij} (Rs./min)
γ_{ij}	tool cost for J_{ij} (Rs./edge)
δ	slack time (min)
λ_{ij}	machining constant for J_{ij}
ϕ	null set
μ	lagrange multiplier
*	shows optimal value

Following additional notations will be used in solution procedure:

D consumed production time (min)

n all nodes belonging to N1

N1 set of all descendents

R set of remaining jobs at any node

s time still available for production (min)

S set of selected jobs at any node

~ shows candidate for acceptance or rejection

[A] is a Guassian notation implying a greatest integer less than or equal to A.

2.4 MATHEMATICAL MODELS AND OPTIMAL SOLUTION PROCEDURES:

Mathematical models for the various cases of the single stage problem stated in Section 2.1 are presented alongwith the optimal solution methodologies.

2.4.1 Case 1: Maximization of Total Units Produced:

In this case we maximize the total amount of production irrespective of the relative values associated with the various jobs to be processed. Two variations of the basic problem are considered. In the first variation we assume that the unit production time is prespecified while in the other case we assume that the production times are dependent upon the machining speeds. The machining speeds are also to be determined optimally considering the workpiece - cutting tool-machine tool considerations.

2.4.1.1 Model 1: Unit Production Time is Prespecified:

Constraints:

Various constraints on the problem are listed below.

i) Constraint on Machine Utilization:

This constraint ensures that total time consumed for production does not exceed the available capacity of the machine. Mathematically, this is represented as:

$$\sum_{i=1}^{N_g} \left(S_i X_i + \sum_{j=1}^{N_i} P_{ij}\, x_{ij}\right) \leq d \tag{2.1}$$

where,

P_{ij} is job production time for producing w_{ij} units of job J_{ij}. It is expressed as,

$$P_{ij} = s_{ij} + w_{ij}\, p_{ij} \tag{2.2}$$

$$x_{ij} = \begin{cases} 1 & \text{if job } J_{ij} \text{ is selected for production} \\ 0 & \text{if job } J_{ij} \text{ is not produced} \end{cases} \tag{2.3}$$

and,

$$X_i = \begin{cases} 1 & \text{if } \sum_{j=1}^{N_i} x_{ij} \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

ii) Constraint on Units Produced:

The quantity w_{ij} to be produced of job J_{ij} should not exceed the lot size specified for it. Thus,

$$0 \leq w_{ij} \leq l_{ij} \qquad \forall\ J_{ij} \in U \tag{2.5}$$

Objective function:

Since the objective is to maximize the total number of the units produced, we maximize a function presented as:

$$Z = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}\, x_{ij} \tag{2.6}$$

Solution procedure:

Let Q represent the time needed to produce all the jobs to their lot sizes. Then,

$$Q = \sum_{i=1}^{N_g} \{S_i + \sum_{j=1}^{N_i} (s_{ij} + l_{ij}\, p_{ij})\} \tag{2.7}$$

If $Q \leq d$, then all the jobs will be produced equal to their lot sizes and the optimal solution will be:

$$x^*_{ij} = 1 \qquad \forall\ J_{ij} \in U \tag{2.8}$$

$$w^*_{ij} = l_{ij} \qquad \forall\ J_{ij} \in U \tag{2.9}$$

$$Z^* = Z_{max} \tag{2.10}$$

where,

$$Z_{max} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} l_{ij} \tag{2.11}$$

If $Q > d$, then some of the jobs are either not accepted for production at all or are not produced to their lot sizes. In such a situation, obviously,

$$Z^* < Z_{max}$$

A branch and bound procedure is developed to optimally ascertain the machine loading and product mix for such a situation.

In the branch and bound procedure, at any stage we branch a node $\tilde{n}$ for which the upper bound value $\bar{Z}$, is maximum among all descendents. $\bar{Z}$ is calculated as the sum of the lower bound value $\underline{Z}$ and maximum of $[s/\{\frac{S_i(1-X_i)+s_{ij}}{l_{ij}} + p_{ij}\}]$ among all remaining jobs at that node, where s shows the remaining time available for production. Initially, $\underline{Z} = 0$. If at this branching job $J_{\tilde{i}\tilde{j}}$ is selected for production, the lower bound value $\underline{Z}$, is incremented by $w_{\tilde{i}\tilde{j}}$.

Thus, the branching procedure finds the lower bound and the upper bound values for each of the descendents. The node with maximum upper bound value is identified. For this node, if the upper bound value is equal to the lower bound value, then branching procedure is terminated and this node carrying the maximum upper bound value gives the optimal solution.

Optimizing algorithm:

The following is the step by step procedure for determining the optimal machine loading and product mix.

Step 1: Calculate Q, from (2.7). If $Q \leq d$, then go to Step 2; otherwise, go to Step 3.

Step 2: Manufacture all the jobs to their lot sizes. The optimal solution is given by the following relationships.

$x^*_{ij} = 1$, $w^*_{ij} = l_{ij}$ $\forall J_{ij} \varepsilon U$ and $Z^* = Z_{max}$.

Terminate the procedure.

Step 3: Follow Branch and Bound Procedure.

Select an initial node; set $n = 1$, $N1 = \{n\}$. For this node n, set $R = U$, $S = \emptyset$, $s = d$, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$. Find a job $J_{\bar{i}\bar{j}}$ from the jobs $J_{ij} \varepsilon R$ for which $(S_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}$ is minimum. Find $\bar{Z} = [d/\{(S_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$ and go to Step 4.

Step 4: Find a node from the nodes $\tilde{n} \varepsilon N1$ for which $\bar{Z}$ is maximum. Denote this node by $\tilde{n}$ and $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. For the node $\tilde{n}$, if $\bar{Z} = \underline{Z}$, go to Step 8; otherwise, go to Step 5.

Step 5: Find $\tilde{D} = D + S_{\tilde{i}}X_{\tilde{i}} + s_{\tilde{i}\tilde{j}} + w_{\tilde{i}\tilde{j}}\, p_{\tilde{i}\tilde{j}}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$; w_{ij} is varied from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D} \leq d$. If $w_{\tilde{i}\tilde{j}} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: Remove the job $J_{\tilde{i}\tilde{j}}$ from the set R of the node $\tilde{n}$. Set $R = R - \{J_{\tilde{i}\tilde{j}}\}$. If $R = \emptyset$, set $\bar{Z} = \underline{Z}$; otherwise, find

$$\min_{J_{ij} \varepsilon R} \left(\frac{S_i X_i + s_{ij}}{l_{ij}} + p_{ij} \right)$$ and denote this job by $J_{\bar{i}\bar{j}}$.

Set $\bar{Z} = \underline{Z} + [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$.

Return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes:

First node: Set $n = n+1$ and $N1 = N1 + \{n\} - \{\tilde{n}\}$. Then for the newly created node n, $R = R - \{J_{\tilde{i}\tilde{j}}\}$, $S = S$, $D = D$, $s = s$, and $\underline{Z} = \underline{Z}$. If $R = \emptyset$, set $\bar{Z} = \underline{Z}$; otherwise,

find $\min_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}\}$ and denote the corresponding job by $J_{\bar{i}\bar{j}}$. Set $\bar{\bar{Z}} = \underline{Z} + [s/\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}} + p_{\bar{i}\bar{j}}\}]$.

Second node: Set n = n+1 and N1 = N1 + {n}. Then for the newly created node n, R = R, $S = S + \{J_{\tilde{i}\tilde{j}}\}$, $D = \tilde{D}$, $s = d - \tilde{D}$, $X_{\tilde{i}} = 0$ and $\underline{\underline{Z}} = \underline{Z} + w_{\tilde{i}\tilde{j}}$. If $R = \phi$, set $\bar{\bar{Z}} = \underline{Z}$; otherwise, find $\min_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}\}$. Denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{\bar{Z}} = \underline{Z} + [s/\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$.

Return to Step 4.

Step 8: Optimal solution will be contained in node $\tilde{n}$, Set, $S^* = S$, $Z^* = \underline{Z}$ and stop.

The value of $w_{\tilde{i}\tilde{j}}$ required in Step 5 is determined using the following relationship.

$$w_{\tilde{i}\tilde{j}} = \left[\frac{d - D - S_{\tilde{i}} X_{\tilde{i}} - s_{\tilde{i}\tilde{j}}}{p_{\tilde{i}\tilde{j}}} \right]$$

If $w_{\tilde{i}\tilde{j}} > l_{\tilde{i}\tilde{j}}$, we set $w_{\tilde{i}\tilde{j}} = l_{\tilde{i}\tilde{j}}$.

Numerical example:

The example given is based on the data of a single stage problem considered by Hitomi and Ham [7].

Consider a single stage problem where ten jobs are available for processing. These jobs are divided into four groups. The relevant data for the problem are given in Table 2.

Table 2.1: Basic Data for Numerical Example 1.

GROUP NO.	PART NO.	LOT SIZE pcs	GROUP SETUP TIME (min)	JOB SETUP TIME (min/lot)	UNIT PRODUCTION TIME (min/pc)
1	1	30	40	19	6
	2	40		8	2
2	1	20	35	10	17
	2	50		9	9
	3	30		15	7
3	1	20	20	5	12
	2	60		13	6
4	1	20	45	6	16
	2	10		10	15
	3	40		20	13

Available Time d = 40 hours.

Solution: Using the various steps of the algorithms, we obtain,

Step 1: $Q = 3105$ mins.

Since $Q > d$, go to Step 3.

Step 3: $n = 1$, $N1 = \{n\} = \{1\}$. For node 1, set

$R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$,

$S = \phi$, $s = 2400.00$ mins., $D = 0$, $\underline{Z} = 0$ and $X_i = 1 \ \forall\ i$.

Calculate $\{(S_i + s_{ij})/l_{ij} + p_{ij}\}$ for $J_{ij} \in R$. The minimum value of 6.55 min/pc corresponds to job J_{32}.

Thus, $\bar{\bar{Z}} = 366$. Proceed to Step 4.

<u>Step 4</u>: Since for node 1, $\bar{\bar{Z}} \neq \underline{Z}$, go to Step 5.

<u>Step 5</u>: For node 1, $\tilde{D} = 393$ mins. and $w_{32} = 60$. Therefore, go to Step 7.

<u>Step 7</u>: Branch out from node 1, two nodes, 2 and 3.

At node 2: Set $n = 2$, $N1 = \{1\} + \{2\} - \{1\} = \{2\}$.

For the newly created node 2, set $R = R - \{J_{32}\}$

$= \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$,

$s = 2400$ mins., $D = 0$ and $S = \emptyset$. Since $R \neq \emptyset$,

calculate $(S_i X_i + s_{ij})/l_{ij} + p_{ij}$ for $J_{ij} \varepsilon R$.

Identify the job for which the value of this expression is minimum. For the present problem the minimum value of the expression corresponds to job J_{12}.

Similarly, at node 3: Set $n = 3$, $N1 = \{2\} + \{3\} = \{2,3$

For the newly created node 3, set

$R = R = \{J_{11}, J_{12}, J_{21} + J_{22}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$,

$D = 393$ mins., $s = 2007$ mins, $S = \{J_{32}\}$, $\underline{Z} = 60$ and

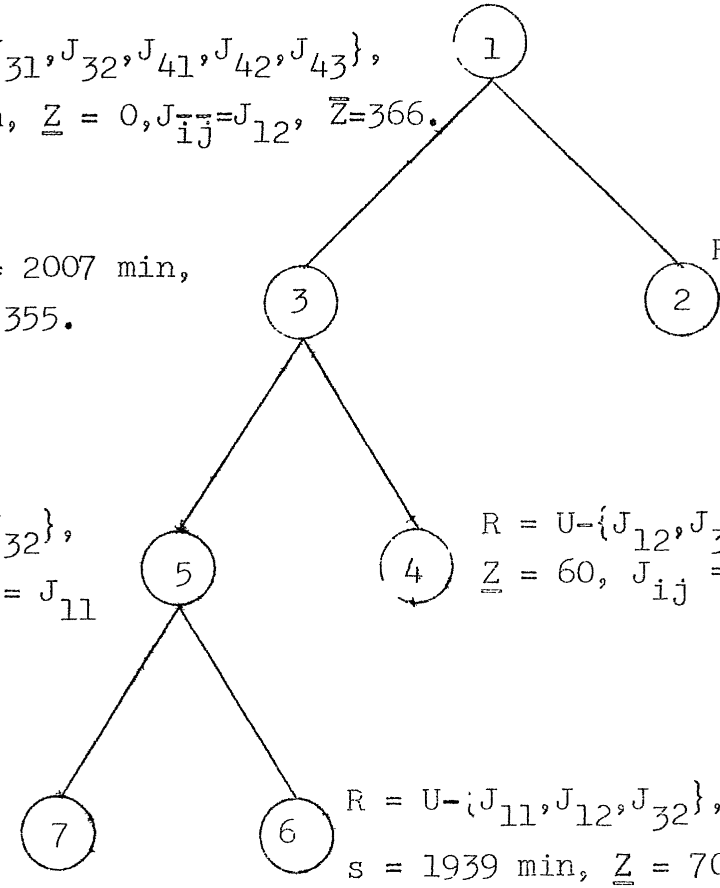
$X_3 = 0$. Since $R \neq 0$, we find the minimum of

$(S_i X_i + s_{ij})/l_{ij} + p_{ij}$. Its value is 6.8 min/pc

corresponding to job J_{12}. We find that $\bar{\bar{Z}} = 355$.

Return to Step 4.

From Step 4, we find that $\max_{n \varepsilon N1} \bar{\bar{Z}}$ occurs at node 3. Further branching takes place at node 3. The steps 4 to 7 of the algorithm are repeated till the branching procedure is fathomed. A portion of the tree-diagram obtained using the



Node 1: $U = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$,
$R = U$, $S = \phi$, $s = 2400$ min, $\underline{Z} = 0$, $J_{\overline{ij}} = J_{12}$, $\overline{Z} = 366$.

Node 2: $R = U - \{J_{32}\}$, $S = \phi$, $s = 2400$ min,
$\underline{Z} = 0$, $J_{\overline{ij}} = J_{12}$, and $\overline{Z} = 352$.

Node 3: $R = U - \{J_{32}\}$, $S = \{J_{32}\}$, $s = 2007$ min,
$\underline{Z} = 60$, $J_{\overline{ij}} = J_{12}$ and $\overline{Z} = 355$.

Node 4: $R = U - \{J_{12}, J_{32}\}$, $S = \{J_{32}\}$, $s = 2007$ min,
$\underline{Z} = 60$, $J_{ij} = J_{11}$ and $\overline{Z} = 311$.

Node 5: $R = U - \{J_{12}, J_{32}\}$, $S = \{J_{12}, J_{32}\}$,
$s = 1939$ min, $\underline{Z} = 70$, $J_{\overline{ij}} = J_{11}$
and $\overline{Z} = 362$.

Node 6: $R = U - \{J_{11}, J_{12}, J_{32}\}$, $S = \{J_{12}, J_{32}\}$,
$s = 1939$ min, $\underline{Z} = 70$, $J_{\overline{ij}} = J_{23}$ and $\overline{Z} = 293$.

Node 7: $R = U - \{J_{11}, J_{12}, J_{32}\}$,
$S = \{J_{11}, J_{12}, J_{32}\}$, $\underline{Z} = 100$,
$s = 1740$ min, $J_{\overline{ij}} = J_{23}$
and $\overline{Z} = 300$.

Fig. 2.1: A portion of the Tree-Diagram obtained for Numerical Example 1 using Optimizing Algorithm.

branching scheme for this example is depicted in Fig. 2.1. The optimal obtained is given in Table 2.2.

Table 2.2: Optimal Solution for Numerical Example 1.

Index of Accepted Group	Index of Accepted Part	Quantity to be produced
1	1	30
	2	10
2	2	50
	3	30
3	1	20
	2	60
4	2	7
	3	40

Total number of units produced = 247

Total time utilized = 2390 mins.

2.4.1.2 Model 2: Machining Speeds are Decision Variables:

Before structuring a mathematical model of the problem, we develop mathematical expressions for job production time and cost of production.

Job Production Time:

The unit production time $p_{ij}(v_{ij})$ for job J_{ij} is a function of machining speed v_{ij} for that job and is given by,

$$p_{ij}(v_{ij}) = a_{ij} + t_{ij} + b_{ij} \frac{t_{ij}}{T_{ij}}$$

or,

$$p_{ij}(v_{ij}) = a_{ij} + \frac{\lambda_{ij}}{v_{ij}} + \frac{\lambda_{ij}\, b_{ij}\, v_{ij}^{(1/n_{ij} - 1)}}{C_{ij}^{1/n_{ij}}}$$

$$\forall \quad J_{ij} \in U \qquad (2.12)$$

where a_{ij} is the preparation time, b_{ij} is the tool replacement time, t_{ij} is actual machining time, T_{ij} is the tool life, λ_{ij} is a machining constant, n_{ij} and C_{ij} are parameters for the Taylor tool life equation for job J_{ij}.

Let the unit production time given by (2.12) be minimum corresponding to a point $v_{ij}^{(t)}$. This machining speed referred to as the 'Maximum-Production-Rate Speed' or 'Minimum-Production-Time Speed', is obtained by setting first derivative of $p_{ij}(v_{ij})$ with respect to v_{ij} equal to zero. This yields,

$$v_{ij}^{(t)} = C_{ij}/\{b_{ij}(1/n_{ij}-1)\}^{n_{ij}} \quad \forall \quad J_{ij} \in U \qquad (2.13)$$

If w_{ij} units of J_{ij} are produced at a speed v_{ij}, the job production time, P_{ij}, is given by,

$$P_{ij} = s_{ij} + w_{ij}\, p_{ij}(v_{ij}) \qquad (2.14)$$

Production Cost:

The per unit production cost, $q_{ij}(v_{ij})$, is also a function of machining speed v_{ij} and is expressed as:

$$q_{ij}(v_{ij}) = \alpha\, a_{ij} + (\alpha + \beta_{ij})\, t_{ij} + (\alpha b_{ij} + \gamma_{ij}) \frac{t_{ij}}{T_{ij}}$$

or,

$$q_{ij}(v_{ij}) = \alpha a_{ij} + (\alpha+\beta_{ij}) \frac{\lambda_{ij}}{v_{ij}} + (\alpha b_{ij} + \gamma_{ij}) \frac{\lambda_{ij} v_{ij}^{(1/n_{ij}-1)}}{C_{ij}^{1/n_{ij}}}$$

$$\forall \; J_{ij} \in U \qquad (2.15)$$

where α is the direct labour cost, β_{ij} is the machining overhead and γ_{ij} is tool cost, respectively.

Let $v_{ij}^{(c)}$ represent the machining speed corresponding to the minimum cost point. The value of this point is obtained by setting the first derivative of (2.15) with respect to v_{ij} equal to zero. The cutting speed, $v_{ij}^{(c)}$, is called the 'Minimum-Production-Cost Machining Speed' and is expressed as follows:

$$v_{ij}^{(c)} = C_{ij} \left\{ \frac{(\alpha + \beta_{ij})}{(1/n_{ij}-1)(\alpha b_{ij} + \gamma_{ij})} \right\}^{n_{ij}}$$

$$\forall \; J_{ij} \in U \qquad (2.16)$$

<u>Constraint:</u>

The various constraints for this model are the same as that for Model 1. The constraint set is given by (2.1) and (2.5).

Objective Function:

In this model we have two different objectives to consider. The first objective is to maximize the total number of units produced while the second objective is to minimize the total cost of production. These objectives are referred as the primary and secondary objectives, respectively. The expression for the total number of units produced is the same as in Model 1 and is given by (2.6). The expression for the cost of production Y, is given as:

$$Y = \sum_{i=1}^{N_g} \{\alpha\ S_i X_i + \sum_{j=1}^{N_g} (\alpha\beta_{ij} + w_{ij}\ q_{ij}(v_{ij}))\ x_{ij}\} \quad (2.17$$

Optimization is carried out using the solution procedure which initially maximizes the primary objective function and then it minimizes the secondary objective function.

Analysis:

Initially for all jobs J_{ij} ε U, we calculate the minimum-production-time machining speed, $v_{ij}^{(t)}$, from (2.13) and the unit production time is then calculated from (2.12). The following expression is used to calculate the job production time for each job considering that each unit in the lot of that job is produced at minimum unit production time:

$$P_{ij}^{(t)} = s_{ij} + l_{ij}\ p_{ij}\ (v_{ij}^{(t)}) \quad (2.18)$$

Let Q^t represent the total time needed to produce all the jobs to their lot sizes. Note that each unit of the lot is produced at the minimum-production-time machining speed. Mathematically,

$$Q^t = \sum_{i=1}^{N_g} (S_i + \sum_{j=1}^{N_i} P_{ij}^{(t)}) \tag{2.19}$$

Depending upon the value of Q^t, there are three possible situations. These are: $Q^t = d$, $Q^t > d$ and $Q^t < d$. These three cases are discussed separately.

i) $\underline{Q^t = d}$: If the total time Q^t, needed to produce all the units is equal to the available capacity d, of the machine, then all the jobs to their lot sizes are produced at the minimum-production-time machining speeds. Therefore, the optimal product mix and the cutting parameters will be given by the following relationships:

$$Z^* = \sum_{J_{ij} \varepsilon U} l_{ij}, \quad x^*_{ij} = 1, \quad v^*_{ij} = v_{ij}^{(t)},$$

$$w^*_{ij} = l_{ij} \qquad \forall\, J_{ij} \;\varepsilon\; U \tag{2.20}$$

where U is the set of all the jobs available for processing.

ii) $\underline{Q^t > d}$: When $Q^t > d$, obviously, all the units cannot be produced even when minimum-production-time machining speeds are used. Any one of the following situations may result.

1. All the jobs are taken up for production. However, for one of the jobs all the units cannot be produced.

2. All the jobs are not taken up for production. The jobs taken up for production are either produced to their lot sizes or for one of the jobs, the desired lot size cannot be produced.

The optimal product mix and machining speeds can be obtained using the following approach.

First of all, we determine the optimal machine loading and product mix based on the minimum-production-time speed for each job using the solution methodology discussed in Section 2.4.1.1 for Model 1. Let S^* and w^*_{ij} ($J_{ij} \in S^*$) respectively represent the set of jobs accepted for production and the quantities of each job to be produced. Then, the optimal solution turns out to be the following:

$$x^*_{ij} = \begin{cases} 1 & \forall J_{ij} \in S^* \\ 0 & \text{Otherwise} \end{cases} \tag{2.21}$$

Clearly this results in,

$$X^*_i = \begin{cases} 0 & \text{If } \sum_{j=1}^{N_i} x^*_{ij} = 0 \\ 1 & \text{Otherwise} \end{cases} \tag{2.22}$$

The total time consumed Cu and the slack time δ are given by the following expressions.

$$Cu = \sum_{i=1}^{N_g} \{S_i X^*_i + \sum_{j=1}^{N_i} (s_{ij} + w^*_{ij} \; p_{ij}{}^{(t)}) \; x^*_{ij}\} \tag{2.23}$$

and $$\delta = d - Cu \tag{2.24}$$

If slack time $\delta = 0$, i.e. if the machine is utilized to its full capacity, then all the jobs selected are produced at the minimum-production-time machining speeds. Obviously,

$$v_{ij}^{*} = v_{ij}^{(t)} \qquad \forall J_{ij} \in S^{*} \tag{2.25}$$

If $\delta > 0$, then we have positive slack time which can be advantageously utilized to minimize the total cost of production. It is obvious that the minimum-production-time machining speed will be higher than the minimum-production-cost machining speed and the cost of production at the minimum-production-time machining speed will be higher than that at minimum-production-cost machining speed. So, it is reasonable to assume that the unit production cost is least at the minimum-production-cost machining speed and it increases with an increase in the machining speed upto the minimum-production-time machining speed. Therefore, the slack time can be utilized in lowering down the cost of production by decreasing the machining speed from $v_{ij}^{(t)}$ to $v_{ij}^{(c)}$. The objective is to determine v_{ij}^{*}, $v_{ij}^{(c)} \leq v_{ij}^{*} < v_{ij}^{(t)}$, such that the total cost of production is minimized. The problem can now be formulated as a nonlinear programme given below:

$$\text{Minimize} \quad Y = \sum_{i=1}^{N_g} \{\alpha S_i X_i^{*} + \sum_{j=1}^{N_i} (\alpha s_{ij} + w_{ij}^{*}\, q_{ij}\, (v_{ij}))\, x_{ij}^{*}\} \tag{2.26}$$

subject to,

$$\sum_{i=1}^{N_g} \{S_i X_i^* + \sum_{j=1}^{N_i} (s_{ij} + w_{ij}^* \, p_{ij}(v_{ij}))\, x_{ij}^*\} \leq d \qquad (2.27)$$

The inequality given by (2.27) represents the machine capacity constraint. The Equations (2.26) and (2.27) contain certain constant terms which can be ignored for the purpose of optimization. The optimization problem can be restated as,

$$\text{Minimize } Y' = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* \, q_{ij}(v_{ij})\, x_{ij}^* \qquad (2.28)$$

subject to,

$$\sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* \, p_{ij}(v_{ij})\, x_{ij}^* \leq d_o \qquad (2.29)$$

where,

$$Y' = Y - \sum_{i=1}^{N_g} (\alpha \, S_i \, X_i^* + \sum_{j=1}^{N_i} \alpha \, s_{ij} \, x_{ij}^*) \qquad (2.30)$$

and

$$d_o = d - \sum_{i=1}^{N_g} (S_i X_i^* + \sum_{j=1}^{N_i} s_{ij} \, x_{ij}^*) \qquad (2.31)$$

For the problem represented by (2.28) and (2.29), a Lagrangian function of the following form is obtained,

$$L(v_{ij}, \mu) = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* \, q_{ij}(v_{ij})\, x_{ij}^* + \mu \{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* \, p_{ij}(v_{ij})\, x_{ij}^* + k^2 - d_o \} \qquad (2.32)$$

where μ and k represent the Lagrangian multiplier and slack variable, respectively. For optimal values of v_{ij}^*, μ^* and k^*, we set $\partial L/\partial v_{ij} = 0$. It gives,

$$\frac{\partial q_{ij}(v_{ij}^*)}{\partial v_{ij}^*} + \frac{\partial p_{ij}(v_{ij}^*)}{\partial v_{ij}^*} = 0$$

or, $$-\left(\frac{\alpha+\beta_{ij}}{v_{ij}^{*2}}\right)\lambda_{ij} + (\alpha\, b_{ij} + \gamma_{ij})\lambda_{ij}\,\frac{v_{ij}^{*\,(1/n_{ij}-2)}}{C_{ij}^{1/n_{ij}}}$$

$$+ \mu^*\left(-\frac{\lambda_{ij}}{v_{ij}} + \frac{\lambda_{ij}\, b_{ij}\, v_{ij}^{*\,(1/n_{ij}-2)}}{C_{ij}^{1/n_{ij}}}\right) = 0$$

From the above equation, we get,

$$\mu^* = \frac{\beta_{ij} - \left(\frac{1}{n_{ij}} - 1\right)\gamma_{ij}\left(\frac{v_{ij}^*}{C_{ij}}\right)^{1/n_{ij}}}{-1 + \left(\frac{1}{n_{ij}} - 1\right) b_{ij}\left(\frac{v_{ij}^*}{C_{ij}}\right)^{1/n_{ij}}} - \alpha \qquad (2.33)$$

and $$v_{ij}^* = C_{ij}\left\{\frac{\alpha + \mu^* + \beta_{ij}}{(\alpha+\mu^* b_{ij}+\gamma_{ij})(1/n_{ij}-1)}\right\}^{n_{ij}} \quad \forall\; J_{ij} \in S^* \qquad (2.34)$$

Similarly, setting,

$$\frac{\partial L}{\partial \mu} = 0 \text{ yields,}$$

$$k^{*2} = d_o - \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^*\, p_{ij}(v_{ij}^*)\, x_{ij}^* \qquad (2.35)$$

and $\partial L/\partial k = 0$ gives,

$$\mu^* k^* = 0 \qquad (2.36)$$

If we substitute $v_{ij}^* = v_{ij}^{(c)}$ in (2.33), we find that $\mu^* = 0$. Similarly, if we substitute $v_{ij}^* = v_{ij}^{(t)}$ in (2.33), we obtain, $\mu^* = \infty$. Hence, it is obvious that,

$$\mu^* \geq 0 \qquad \text{for } v^*_{ij} \in (v_{ij}^{(c)}, v_{ij}^{(t)}) \qquad (2.37)$$

Since μ^* is nonnegative, it guarantees the minimization of total production cost.

If $\mu^* = 0$, then from (2.36), $k^* \geq 0$ and from (2.34),

$$v^*_{ij} = v_{ij}^{(c)} \qquad \forall J_{ij} \in S^* \qquad (2.38)$$

However, if $\mu^* > 0$, then from (2.36) we know that $k^* = 0$. The optimal values of μ^* and $v^*_{ij} \in E_{ij}$ can now be determined from (2.33) and (2.34), respectively, by using hit and trial approach. E_{ij} is called the high-efficiency speed range. Mathematically,

$$E_{ij} = [v_{ij}^{(c)}, v_{ij}^{(t)}] \qquad (2.39)$$

iii) $\underline{Q^t < d}$: When $Q^t < d$, the optimal solution turns out to be the following:

$$R^* = \phi, \; S^* = U, \; X^*_i = 1 \quad \forall\; i, \; Z^* = \sum_{J_{ij} \in S^*} l_{ij},$$

$$x^*_{ij} = 1 \text{ and } w^*_{ij} = l_{ij} \quad \forall\; J_{ij} \in S^* \qquad (2.40)$$

In this situation the slack time is expressed as,

$$\delta = d - Q^t \qquad (2.41)$$

We can minimize the total cost of production by utilizing the available slack time. Once the available slack time is obtained, the optimal machining speeds can be determined by formulating a nonlinear programme of the problem given by (2.26) and (2.27). The optimal machining speeds will be given by (2.34).

Optimizing Algorithm:

A step-by-step algorithm, for finding the optimal product mix and machine loading when the machining speeds are also decision variables, is given below:

Step 1: For each job $J_{ij} \varepsilon U$ find $v_{ij}^{(t)}$ and $p_{ij}^{(t)}$ using (2.13) and (2.12), respectively. From (2.18) and (2.19), find Q^t. If $Q^t \leq d$, go to Step 2; otherwise, go to Step 3.

Step 2: The optimal solution is,

$$S^* = U$$

$$x^*_{ij} = 1 \qquad \forall \; J_{ij} \varepsilon S^*$$

$$w^*_{ij} = l_{ij} \qquad \forall \; J_{ij} \varepsilon S^*$$

$$X^*_i = 1 \qquad \forall \; i$$

$$Z^* = \sum_{J_{ij} \varepsilon S^*} l_{ij}$$

If $Q^t = d$, set $v^*_{ij} = v_{ij}^{(t)} \; \forall \; J_{ij} \; \varepsilon \; S^*$ and stop. For $Q^t < d$, go to Step 9.

Step 3: Follow branch and bound procedure. Select an initial node: set $n = 1$, $N1 = \{n\}$. For this node, Set $R = U$, $S = \phi$, $D = 0$, $s = d$, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$. Find a job $J_{\bar{i}\bar{j}}$ from the jobs $J_{ij} \; \varepsilon \; R$, for which $(S_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}$ is minimum.

Set $\bar{Z} = [\; d / \{(S_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$ and go to Step 4.

Step 4: Find a node $\tilde{n}$ from the nodes $n \varepsilon N1$, for which $\bar{Z}$ is maximum. Denote the job $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. For the node $\tilde{n}$, if $\underline{Z} = \bar{Z}$, go to Step 8; otherwise, go to Step 5.

Step 5: Find $\tilde{D} = D + S_{\tilde{i}} X_{\tilde{i}} + s_{\tilde{i}\tilde{j}} + w_{\tilde{i}\tilde{j}} p_{\tilde{i}\tilde{j}}^{(t)}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$; w_{ij} is varied from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D} \leq d$. If $w_{\tilde{i}\tilde{j}} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: Remove the job $J_{\tilde{i}\tilde{j}}$ from the list R. Set $R = R-\{J_{\tilde{i}\tilde{j}}\}$. If $R = \phi$, set $\bar{Z} = \underline{Z}$. Otherwise, find

$\min\limits_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}^{(t)}\}$, denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{Z} = \underline{Z} + [s/\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$.

Return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes:

At first node: $n = n+1$, $N1 = N1 + \{n\} - \{\tilde{n}\}$.

For node n, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$, $S = S$, $D = D$, $s = s$ and $\underline{Z} = \underline{Z}$. If $R = \phi$, set $\bar{Z} = \underline{Z}$. Otherwise, find

$\min\limits_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}^{(t)}\}$. Denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{Z} = \underline{Z} + [s/\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$. Similarly at second node: $n = n+1$, $N1 = N1 + \{n\}$. For this node n, set $R = R$, $S = S + \{J_{\tilde{i}\tilde{j}}\}$, $D = \tilde{D}$, $s = d - D$, $X_{\tilde{i}} = 0$ and $\underline{Z} = \underline{Z} + w_{\tilde{i}\tilde{j}}$. If $R = \phi$, set $\underline{Z} = \bar{Z}$. Otherwise, find $\min\limits_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}^{(t)}\}$.

Denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set

$\bar{Z} = \underline{Z} + [s/\{(S_{\bar{I}}X_{\bar{I}} + s_{\bar{i}\bar{j}})/1_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$.

Return to Step 4.

Step 8: The optimal solution given by list $\tilde{n}$, is,

$$S^* = S$$

$$x_{ij}^* = \begin{cases} 1 & \forall \ J_{ij} \in S^* \\ 0 & \text{Otherwise} \end{cases}$$

$$Z^* = \underline{Z}$$

$$X_i^* = \begin{cases} 0 & \text{if } \sum_{j=1}^{N_i} x_{ij}^* = 0 \\ 1 & \text{otherwise} \end{cases}$$

Go to Step 9.

Step 9: Find $v_{ij}^{(c)}$ from (2.16) and $p_{ij}^{(c)}$ from (2.12), respectively. Calculate

$$Q = \sum_{i=1}^{N_g} \{S_i X_i^* + \sum_{j=1}^{N_i} (s_{ij} + w_{ij}^* \, p_{ij}^{(c)}) \, x_{ij}^* \}$$

If $Q \leq d$, set $v_{ij}^* = v_{ij}^{(c)} \ \forall \ J_{ij} \in S^*$ and stop.

Otherwise, go to Step 10.

Step 10: Select a job $J_{ij} \in S^*$. Find initial value of μ from the following expression,

$$\mu = \frac{\beta_{ij} - (\frac{1}{n_{ij}} - 1) \, \gamma_{ij} \left(\frac{v_{ij}^{(t)} + v_{ij}^{(c)}}{2C_{ij}} \right)^{1/n_{ij}}}{-1 + (\frac{1}{n_{ij}} - 1) \, b_{ij} \left(\frac{v_{ij}^{(t)} + v_{ij}^{(c)}}{2C_{ij}} \right)^{1/n_{ij}}} - \alpha$$

Go to Step 11.

Step 11: For the value of μ obtained in Step 10, find $v_{ij}^{(e)}$ and p_{ij} from (2.34) and (2.12), respectively. Calculate,

$$Q = \sum_{i=1}^{N_g} \{S_i X_i^* + \sum_{j=1}^{N_i} (s_{ij} + w_{ij}^* p_{ij}) x_{ij}^* \}$$

If $Q \neq d$, go to Step 12; otherwise, stop. The set of machining speeds obtained currently is optimal.

Step 12: Modify the value of μ appropriately depending on whether $Q < d$ or $Q > d$. If $Q < d$, reduce the value of μ and if $Q > d$ increase the value of μ. Go to Step 11.

A systematic procedure for the selection of the value of μ is presented in the form of a block-diagram in Fig. 2.2.

Numerical Example 2:

For better understanding of the solution procedure, consider the single stage problem considered by Hitomi and Ham [8]. In all, there are ten jobs available for processing. These jobs are divided into 4 groups. The basic data are given in Table 2.3.

Solution:

Step 1: Initially, the values of $v_{ij}^{(t)}$ and $p_{ij}^{(t)}$ are found for all jobs $J_{ij} \in U$. These are shown in Table 2.4. We obtain, Q^t = 5830.574 mins. Since $Q^t > d$, go to Step 3.

Step 3: Use branch and bound procedure.

Set $n = 1$, $N1 = \{n\} = \{1\}$.

For node 1, we get, $R = U$, $S = \emptyset$, $D = 0$, s = 3000.00 mins.

$\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$.

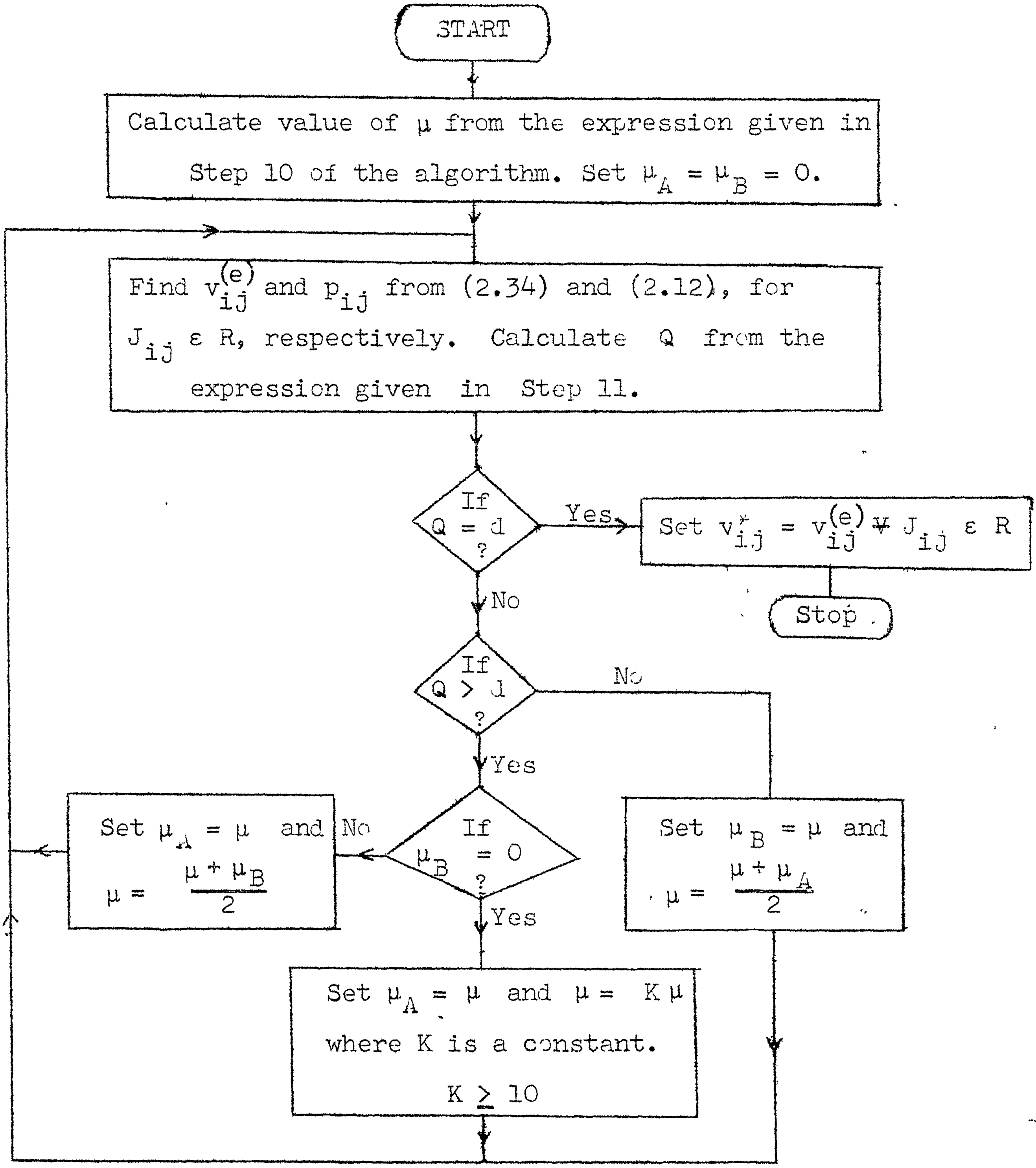


Fig. 2.2: Block-diagram for determination of μ^* and v_{ij}^* .

Table 2.3: **Basic data for the numerical Example**

GROUP	PART	LOT SIZE	MACHINING CONSTANT	TOOL-LIFE PARAMETERS: SLOPE CONSTANT	TOOL-LIFE PARAMETERS: 1-MIN LIFE MACHINING SPEED	TIME PARAMETERS: GROUP SETUP TIME	TIME PARAMETERS: LOT SETUP TIME	TIME PARAMETERS: PREPARATION TIME	TIME PARAMETERS: TOOL REPLACEMENT TIME	COST PARAMETERS: DIRECT LABOR COST AND OVERHEAD	COST PARAMETERS: MACHINING OVERHEAD	COST PARAMETERS: TOOL COST
i	j	l_{ij}	λ_{ij}	n_{ij}	C_{ij}	S_i	s_{ij}	a_{ij}	b_{ij}	α_{ij}	β_{ij}	γ_{ij}
No.	No.	pcs			m/min	min	min/lot	min/pc	min/edge	c/min	c/min	c/edge
1	1	60	707	0.25	350	20.00	19.00	2.50	2.00	15	15	500
	2	50	377	0.25	350		8.00	3.00	3.50	15	25	450
2	1	100	1257	0.33	400	22.00	10.00	3.00	3.00	15	15	600
	2	70	982	0.25	250		9.00	4.00	2.50	15	20	800
	3	40	565	0.20	240		15.00	2.50	2.50	15	15	450
3	1	30	565	0.25	250	15.00	5.00	3.00	3.00	15	30	600
	2	90	626	0.33	300		13.00	5.00	1.50	15	15	500
4	1	40	424	0.20	350	25.00	6.00	4.00	3.50	15	20	450
	2	50	1257	0.25	350		10.00	3.00	4.00	15	20	550
	3	80	524	0.20	200		20.00	2.50	4.50	15	25	600

Total available production time = 50 hrs.

Table 2.4: Maximum Production Rate Machining Speed and Unit Production Time.

Index of Group	Index of Part	Machining Speed m/min.	Unit Production Time min/pc
1	1	223.6301	6.715
	2	194.4333	5.585
2	1	220.3509	11.514
	2	151.0688	12.667
	3	151.4298	7.164
3	1	144.3376	8.219
	2	207.7381	9.498
4	1	206.4631	6.567
	2	188.0497	11.912
	3	112.1955	8.338

Determine the values of the expression $(S_i+s_{ij})/l_{ij} + p_{ij}(t$ $\forall\, J_{ij} \in R$ and identify the job $J_{\overline{ij}}$ for which the value of the expression is minimum. For the present problem, $J_{\overline{ij}}$ is J_{12}, and the value of expression is 6.195 min/pc.

Step 4: The maximum value of the upper bound, $\overline{Z}$ corresponds to node 1. Since $\overline{Z} \neq \underline{Z}$, go to Step 5.

Step 5: For node 1, we find $w_{12} = 50$ and $\tilde{D} = 307.25$ mins. Since $w_{12} > 0$, go to Step 7.

Step 7: Branch the node 1 into nodes 2 and 3, and return to Step 4.

At node 2: $n = n+1 = 2$, $N1 = \{1\} + \{2\} - \{1\} = \{2\}$.

Set $R = \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \phi$, $s = 3000.00$ mins, $D = 0$ and $\underline{Z} = 0$. Since $R \neq \phi$, determine $\{(S_i+s_{ij})/l_{ij} + p_{ij}^{(c)}\}$ for $J_{ij} \varepsilon R$. Identify the job $J_{\overline{ij}}$ for which the value of the expression is minimum. For the present problem , $J_{\overline{ij}}$ is J_{41}, and value of expression is 7.342min/pc. Therefore, $\underline{Z} = 408$.

At node 3: $n = n+1 = 3$, $N1 = \{2\} + \{3\} = \{2,3\}$.

Set $R = \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{12}\}$, $s = 2692.75$ min, $D = 307.25$ min., $\underline{Z} = 50$ and $X_1 = 0$. Since $R \neq \phi$, determine $\{(S_i + s_{ij})/l_{ij} + p_{ij}^{(c)}\}$ for $J_{ij} \varepsilon R$. Indentify the job $J_{\overline{ij}}$ for which the expression carries minimum value. Find that $J_{\overline{ij}}$ is J_{11} for which the value of the expression is 7.032 min/pc. Set, $\overline{\overline{Z}} = 432$.

This time from the Step 4, we find that $\max_{n \varepsilon N1} \overline{\overline{Z}}$ occurs at node 2. Further branching is done from node 2 following the steps 4 to 7 of the algorithm. Table 2.5 gives the optimal results obtained.

The optimal solution shown in Table 2.5 is compared with the results of the same example obtained by Hitomi and Ham [8]. It is observed that in their case, the total production amount was 356 pcs while in our case it is 372 pcs.

Table 2.5: Optimal Solution for the Numerical Example 2.

Index of Accepted Group	Index of Accepted Part	Optimal Machining Speed m/min.	Unit Production Time min/pc	Quantity to be produced pcs
1	1	173.9853	6.81	60
	2	134.3816	5.53	50
2	3	126.7238	7.14	40
3	1	117.0901	8.06	30
	2	134.0509	10.08	72
4	1	181.4794	6.42	40
	3	99.0188	7.95	80

Total number of units produced = 372

Total cost of production = Rs. 20,998.87

Total time utilized = 2,999.22 mins.

2.4.2 Case 2: Profit Maximization:

The problem of total profit maximization is considered for the following situations:

1. The unit production time is prespecified.
2. The machining speed which ultimately determines the unit production time is a decision variable.

3.4.2.1 Model 3: Unit Production time is prespecified.

The expression for job production time and the constraints for this problem turn out to be the same as for Model 1. The job production time is given by (2.12) while the various constraints are represented by (2.1) and (2.5).

Objective function:

The function representing the total profit generated by the single stage production system under consideration can be written as:

$$Z = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} B_{ij}\, w_{ij}\, x_{ij} \tag{2.42}$$

where B_{ij} is the profit per unit of job J_{ij}, x_{ij} is 0-1 variable and is given by (2.3). The objective is to maximize the total profit, i.e., Z.

Solution Methodology:

In this model also, we initially find Q, i.e., the time required to produce all the units of all the jobs. Q is obtained from (2.7). If $Q \leq d$, all the jobs can be produced to their lot sizes and the optimal solution will be:

$$w_{ij}^{*} = l_{ij} \qquad \forall\ J_{ij} \in U$$

$$Z^{*} = Z_{max}$$

where,

$$Z_{max} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} B_{ij}\, l_{ij} \tag{2.43}$$

However, if $Q > d$, the optimal $Z^* < Z_{max}$. Under this situation, we restore to B & B technique to find the optimal solution.

At any stage of the B & B procedure, we select a node $\tilde{n}$ from all the descendents for which the upper bound value of the objective function, $\bar{\bar{Z}}$, is maximum. Here $\bar{\bar{Z}}$ is calculated as the sum of the lower bound value of the objective function $\underline{Z}$, and $\max\limits_{J_{ij} \varepsilon R} B_{ij} [s/\{(S_i \overline{1-X_i} + s_{ij})/ l_{ij} + p_{ij}\}]$. Note that at the first node, the lower bound value $\underline{Z}$ is zero. However, at the subsequent nodes when a job $J_{\tilde{i}\tilde{j}}$ is selected for production, $\underline{Z}$ is incremented by the quantity $B_{\tilde{i}\tilde{j}} w_{\tilde{i}\tilde{j}}$ for the node where the selected job $J_{\tilde{i}\tilde{j}}$ is assigned. The B & B procedure branches the node $\tilde{n}$ into two nodes, one in which job $J_{\tilde{i}\tilde{j}}$ is in the set of selected jobs while in the other node $J_{\tilde{i}\tilde{j}}$ is not in the list of selected jobs. For each of the descendents, we calculate the lower bound and the upper bound values of the objective function. The B & B procedure is terminated as soon as a descendent with the maximum upper bound value is obtained for which the upper bound and the lower bound values of the objective function are equal. The list of jobs in this node gives the optimal product mix.

Optimizing Algorithm:

Following is the step by step algorithm to find the optimal solution.

Step 1: Calculate Q from (2.7). If $Q \leq d$, go to Step 2; otherwise, go to Step 3.

Step 2: The optimal solution is:

$$S^* = U$$

$$w^*_{ij} = l_{ij} \quad \forall \; J_{ij} \in S^*$$

$$Z^* = Z_{max} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} B_{ij} \, l_{ij}$$

Terminate the procedure. Follow branch and bound procedure.

Step 3: Select an initial node: $n = 1$, $N1 = \{n\}$.
For this node, set $R = U$, $S = \emptyset$, $D = 0$, $s = d$, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$. Find a job $J_{\bar{i}\bar{j}}$ from the jobs $J_{\bar{i}\bar{j}} \in R$, for which $Y_{\bar{i}\bar{j}} = B_{\bar{i}\bar{j}}/\{(S_{\bar{i}} \; X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}})\}$
is maximum. Set $\bar{Z} = B_{\bar{i}\bar{j}} \; [d/\{(S_{\bar{i}} \; X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$ and go to Step 4.

Step 4: Find $\max_{n \varepsilon N1} \bar{Z}$ and denote the corresponding node by $\tilde{n}$ and the job $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. If for the node $\tilde{n}$, $\bar{Z} = \underline{Z}$, go to Step 8; otherwise, go to Step 5.

Step 5: Find $\tilde{D} = D + S_{\tilde{i}}X_{\tilde{i}} + s_{\tilde{i}\tilde{j}} + w_{\tilde{i}\tilde{j}} \, p_{\tilde{i}\tilde{j}}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$, w_{ij} is varied appropriately from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D} \leq d$. If $w_{ij} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: At node $\tilde{n}$, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$. If $R = \emptyset$, set $\bar{Z} = \underline{Z}$; otherwise, find a job $J_{\bar{i}\bar{j}}$ from jobs $J_{ij} \in R$ for which

$Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$. Set $\bar{\bar{Z}} = \underline{\underline{Z}} + B_{\bar{i}\bar{j}} [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}}$
$+ p_{\bar{i}\bar{j}}\}]$ and return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes.

At the first node: Set $n = n+1$, $N1 = N1 + \{n\} - \{\tilde{n}\}$.
Further, for the new node n, set $R = R-\{J_{\tilde{i}\tilde{j}}\}, S = S$,
$s = s$, $D = D$ and $\underline{\underline{Z}} = \underline{\underline{Z}}$. If $R = \phi$, set $\bar{\bar{Z}} = \underline{\underline{Z}}$; otherwise
find a job $J_{\bar{i}\bar{j}}$ for which $Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$.
Set $\bar{\bar{Z}} = \underline{\underline{Z}} + B_{\bar{i}\bar{j}} [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$.

At the second node: Set $n = n+1$, $N1 = N1 + \{n\}$. For the newly created node n, set $R = R$, $S = S + \{J_{\tilde{i}\tilde{j}}\}$, $s = d-\tilde{D}$,
$D = \tilde{D}$, $X_{\tilde{i}} = 0$ and $\underline{\underline{Z}} = \underline{\underline{Z}} + w_{\tilde{i}\tilde{j}} B_{\tilde{i}\tilde{j}}$. If $R = \phi$, set $\bar{\bar{Z}} = \underline{\underline{Z}}$;
otherwise, find a job $J_{\bar{i}\bar{j}}$ for which $Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$.
Set $\bar{\bar{Z}} = \underline{\underline{Z}} + B_{\bar{i}\bar{j}}[s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$.
Return to Step 4.

Step 8: The optimal solution is given by node $\tilde{n}$. Set $S^* = S$,
$Z^* = \underline{\underline{Z}}$ and stop.

Numerical Example 3:

Consider a single stage production system in which ten jobs are available for productions. The jobs are divided into four groups. The basic data for the example are given in Table 2.6.

Solution: The solution is given using the various steps of the algorithm:

Step 1: $Q = 3105$ mins. Since $Q > d$, go to Step 3.

Table 2.6: Basic Data Considered for Numerical Example 3.

Index of Group	Index of Part	Profit per Unit (Rs./pc)	Lot Size (pcs)	Group Set-up Time (min.)	Job Set-up Time (min/lot)	Unit Production Time (min/pc)
1	1	6.0	30	40	19	6
	2	2.0	40		8	2
2	1	17.0	20	35	10	17
	2	9.0	50		9	9
	3	7.0	30		15	7
3	1	12.0	20	20	5	12
	2	6.0	60		13	6
4	1	16.0	20	45	6	16
	2	13.0	10		10	15
	3	15.0	40		20	13

Total Available Production Time = 40 hrs.

Step 3: Follow the branch and bound procedure.

Set $n = 1$, $N1 = \{n\} = \{1\}$.

For the newly created node 1, set

$R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \emptyset$, $\underline{Z} = 0$, $s = d$, $D = 0$ and $X_i = 0 \ \forall\, i$. Calculate Y_{ij} for all $J_{ij} \in R$. Identify the job for which Y_{ij} is maximum. For the present problem Y_{32} is maximum and equal to 0.9160 Rs./min. Therefore, $J_{\bar{i}\bar{j}}$ is J_{32}. Set $\bar{Z} = B_{32}\,[d/\{(S_3 + s_{32})/l_{32} + p_{32}\}] =$ Rs.2196.00 and go to Step 4.

Step 4: $\max\limits_{n \in N1} \bar{Z}$ occurs at node 1. Since, for node 1, $\bar{Z} \neq \underline{Z}$, go to Step 5.

Step 5: $w_{32} = 60$ pcs and $\tilde{D} = 393$ mins. Since $w_{32} > 0$, go to Step 7.

Step 7: Branch the node 1 into two nodes, 2 and 3 and return to Step 4. Various parameters at the nodes, 2 and 3, are given below:

At Node 2: Set $n = n+1 = 2$, $N1 = \{1\} - \{1\} + \{2\} = \{2\}$. For node n, set $R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$, $S = S$, $\underline{Z} = 0$; $s = 2400$ mins. and $D = 0$. Since $R \neq \emptyset$, find Y_{ij} for all $J_{ij} \in R$. Note that $\max\limits_{J_{ij} \in R} Y_{ij}$ is equal to 0.9109 Rs/min and it corresponds to J_{22}. Set $\bar{Z} = \underline{Z} + B_{22}\,[s/\{S_2X_2 + s_{22})/l_{22} + p_{22}\}]$

= Rs. 2178.00

At node 3: Set $n = n+1 = 3$, $N1 \{2\} + \{3\} = \{2,3\}$. For the newly created node n, set
$R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{32}\}$, $s = d - \tilde{D} = 2007$ mins, $D = 393$ mins. and $\underline{Z}$ = Rs. 360.00. Since $R \neq \phi$, find Y_{ij} for all $J_{ij} \varepsilon R$. Identify the job for which Y_{ij} is maximum. $\max_{J_{ij} \varepsilon R} Y_{ij}$ corresponds to job J_{31}. Therefore,

$$\overline{Z} = \underline{Z} + B_{31} [s/\{(S_3 X_3 + s_{31})/l_{31} + p_{31}\}] = \text{Rs. } 2316.0$$

Now we return to Step 4 and find that $\max_{n \varepsilon N1} \overline{Z}$ occurs at node 3. At this node, we observe that the upper and the lower bound values of the objective function are not equal. Therefore, we go to Step 5. Steps 4 to of the algorithm are repeated till the optimality criteria is satisfied. Table 2.7 gives the optimal solution for this example.

Table 2.7: Optimal Solution for the Numerical Example 3.

Index of Accepted Group	Index of Accepted Part	Quantity to be Produced (pcs)	Profit (Rs)
2	1	20	340
	2	50	450
	3	30	210
3	1	20	240
	2	60	360
4	1	1	16
	3	40	600

Total profit = Rs. 2,216.00
Total time utilized = 2,394 mins.

2.4.2.2 Model 4: Machining speeds are decision variables:

For this model, the mathematical expressions for job production time, unit production time and production cost, and the constraints turn out to be the same as for Model 2.

Equation (2.14) represents the job production time while the constraints are given by (2.1) and (2.5). The machining speeds corresponding to minimum-production-time and the minimum-production-cost are given by (2.13) and (2.16).

Objective Function:

Our primary objective is to determine product mix and machining speeds such that the total profit is maximized. However, once the values of optimal decision variables which yield maximum profit are obtained, one can further manipulate the machining speeds such that the cost of production is minimiz and the maximum profit is still realized.

The primary objective function representing the total profit turns out to be the same as given by (2.42). The secondary criteria of minimizing the total cost of production involves the consideration of total production cost expression given by (2.17).

Analysis:

In this model, the analysis is almost same as done for Model 2. The relationships given in (2.18) through (2.41) are also applicable for this model.

Initially, Q^t, is found from (2.9) and then the time needed to produce all the jobs to their lot sizes at their maximum-production-rate machining speeds, is compared with d, the time available for production. Depending upon the values of Q^t and d, one of the following situations may result.

i) $\underline{Q^t = d}$:

If $Q^t = d$, then the optimal solution will be as given by (2.20) and the value of primary objective function Z^* will be equal to Z_{max}.

ii) $\underline{Q^t > d}$:

If $Q^t > d$, then set $p_{ij} = p_{ij}^{(t)}$ for all jobs $J_{ij} \varepsilon U$. The optimal product mix and machine loading is obtained using the solution procedure of Model 3. If the slack time $\delta > 0$, then this available slack time can be advantageously utilized to minimize the total cost of production. In view of this, the modified optimal machining speeds are determined using the solution procedure developed for Model 2.

iii) $\underline{Q^t < d}$:

If $Q^t < d$, then the optimal machining speeds are determined corresponding to the minimum-production-cost. Thus, $v_{ij}^* = v_{ij}^{(e)} \varepsilon E_{ij}$. The values of v_{ij}^* are determined utilizing the slack time using the solution procedure described for Model 2.

Optimizing Algorithm:

Following is the stepwise description of the algorithm:

Step 1: Find $v_{ij}^{(t)}$ and $p_{ij}^{(t)}$ for all the jobs $J_{ij} \in U$ from (2.13) and (2.12), respectively. Calculate Q^t, from (2.18) and (2.19). If condition, $Q^t \leq d$, is satisfied, go to Step 2; otherwise, go to Step 3.

Step 2: The optimal solution is:

$$S^* = U$$

$$X_i^* = 1 \quad \forall\, i$$

$$x_{ij}^* = 1 \quad \forall\, J_{ij} \in S^*$$

$$w_{ij}^* = l_{ij} \quad \forall\, J_{ij} \in S^*$$

$$Z^* = Z_{max} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} B_{ij}\, l_{ij}$$

If $Q^t = d$, set $v_{ij}^* = v_{ij}^{(t)}$ $\forall\, J_{ij} \in S^*$ and stop; otherwise, go to Step 9.

Step 3: Follow branch and bound procedure.

Form the initial node: set $n = 1$, $N1 = \{n\}$.

For the initial node $n = 1$, set $R = U$, $S = \emptyset$, $D = 0$, $s = d$, $\underline{Z} = 0$ and $X_i = 1$ $\forall\, i$. Find a job $J_{\bar{i}\bar{j}}$ from the jobs $J_{ij} \in R$, for which $Y_{\bar{i}\bar{j}} = B_{\bar{i}\bar{j}}/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(}$ is maximum. Set $\bar{Z} = B_{\bar{i}\bar{j}}\,[d/\{(S_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$ and go to Step 4.

Step 4: Find a node $\tilde{n}$ from the nodes $n \in N1$ for which $\bar{Z}$ is maximum. Denote the job $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. If for node $\tilde{n}$, $\bar{Z} = \underline{Z}$, go to Step 8, else go to Step 5.

Step 5: Find $\tilde{D} = D + S_{\tilde{i}} X_{\tilde{i}} + s_{\tilde{i}\tilde{j}} + w_{\tilde{i}\tilde{j}}\, p_{\tilde{i}\tilde{j}}^{(t)}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$; w_{ij} is varied from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D} \leq d$. If $w_{\tilde{i}\tilde{j}} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: In node $\tilde{n}$, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$. If $R = \emptyset$, set $\overline{Z} = \underline{Z}$; otherwise, find a job $J_{\bar{i}\bar{j}}$ for which $Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$. Set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\, [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$. Return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes as given below:

At first node: Set $n = n+1$, $N1 = N1 + \{n\} - \{\tilde{n}\}$. For the newly created node n, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$, $S = S$, $s = s$, $D = D$ and $\underline{Z} = \underline{Z}$. If $R = \emptyset$, set $\overline{Z} = \underline{Z}$; otherwise, find a job $J_{\bar{i}\bar{j}}$ for which $Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$.
Set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\, [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$.

At second node: Set $n = n+1$, $N1 = N1 + \{n\}$. For this new node n, set $R = R$, $S = S + \{J_{\tilde{i}\tilde{j}}\}$, $X_{\tilde{i}} = 0$, $s = d - \tilde{D}$, $D = \tilde{D}$ and $\underline{Z} = \underline{Z} + w_{\tilde{i}\tilde{j}}\, B_{\tilde{i}\tilde{j}}$. If $R = \emptyset$, set $\overline{Z} = \underline{Z}$; otherwise, find a job $J_{\bar{i}\bar{j}}$ for which $Y_{\bar{i}\bar{j}} = \max\limits_{J_{ij}\varepsilon R} Y_{ij}$. Set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\, [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}^{(t)}\}]$. Return to Step 4.

Step 8: The optimal solution is,

$$S^* = S$$

$$x^*_{ij} = \begin{cases} 1 & \forall\, J_{ij}\ \varepsilon\ S^* \\ 0 & \text{Otherwise} \end{cases}$$

$$Z^* = \underline{Z}$$

$$X_i^* = \begin{cases} 0 & \text{If } \sum_{j=1}^{N_i} x_{ij}^* = 0 \ \forall\, i \\ 1 & \text{Otherwise} \end{cases}$$

Go to Step 9.

Step 9: Find $v_{ij}^{(c)}$ and $p_{ij}^{(c)}$, using (2.16) and (2.12), respectively. Calculate Q,

$$Q = \sum_{i=1}^{N_g} \{ S_i X_i^* + \sum_{j=1}^{N_i} (s_{ij} + w_{ij}^* p_{ij}^{(c)}) x_{ij}^* \}$$

If $Q \leq d$, set $v_{ij}^* = v_{ij}^{(c)}$ for $J_{ij} \varepsilon S^*$ and stop; otherwise, go to Step 10.

Step 10: Select a job $J_{ij} \varepsilon S^*$ to determine initial value of μ. Mathematically,

$$\mu = \frac{\beta_{ij} - (\frac{1}{n_{ij}} - 1)\, \gamma_{ij} (\frac{v_{ij}^{(t)} + v_{ij}^{(c)}}{2\, C_{ij}})^{1/n_{ij}}}{-1 + (\frac{1}{n_{ij}} - 1)\, b_{ij}\, (\frac{v_{ij}^{(t)} + v_{ij}^{(c)}}{2 C_{ij}})^{1/n_{ij}}} - \alpha$$

Step 11: Using the value of μ obtained from Step 10, find $v_{ij}^{(e)}$ and p_{ij} for all $J_{ij} \varepsilon S^*$ from (2.34) and (2.12), respectively. Recalculate Q from the following mathematical expression:

$$Q = \sum_{i=1}^{N_g} \{ S_i X_i^* + \sum_{j=1}^{N_i} (s_{ij} + w_{ij}^* p_{ij}) x_{ij}^* \}.$$

If $Q \neq d$, go to Step 12; otherwise, stop. The present set of machining speeds is optimal.

Step 12: Modify the value of μ appropriately depending on whether $Q < d$ or $Q > d$. If $Q < d$, reduce the value of μ and if $Q > d$, decrease the value of μ. Go to Step 11.

Numerical Example 4:

Consider the numerical example 2 alongwith the additional information given in Table 2.8. The objective is to determine optimal product mix and machining speeds to maximize the total profit.

Table 2.8: Basic Datas for Numerical Example 4.

Index of Group	Index of Part	Profit per unit Rs./pc
1	1	10.00
	2	13.00
2	1	8.00
	2	2.00
	3	18.00
3	1	6.50
	2	8.50
4	1	11.00
	2	14.00
	3	5.50

Solution:

Step 1: Determine for all jobs J_{ij} ε U, the values of $v_{ij}^{(t)}$ and $p_{ij}^{(t)}$. These values are given in Table 2.4. Now with present set of $p_{ij}^{(t)}$, Q^t = 5830.574 mins. Since $Q^t > d$, go to Step 3.

Step 2: Follow branch and bound procedure.

Set n = 1, N1 = {n} = {1}.

For node 1, set R = U, S = ϕ, $X_i = 1 \; \forall \; i$, $\underline{Z} = 0$, s = 3000 mins. and D = 0.

Determine for all jobs J_{ij} ε R, the value of Y_{ij} given by expression, $B_{ij}/\{(S_i + s_{ij})/l_{ij} + p_{ij}^{(t)}\}$.

Identify the maximum value of Y_{ij}. For the present problem, $\max\limits_{J_{ij} \varepsilon R} Y_{ij}$ corresponds to job J_{23} and this is equal to 2.225 Rs./min. Therefore, $\bar{Z} = B_{23}[d/\{(S_2+s_{22})/l_{23} + p_{23}^{(t)}\}]$ = Rs. 6678.00. Now go to Step 4.

Step 4: $\max\limits_{n \varepsilon N1} \bar{Z}$ is found in node 1. Its value is Rs. 6678.00. Since $\bar{Z} \neq \underline{Z}$, go to Step 5.

Step 5: $\tilde{D}$ = 323.56 mins. and w_{23} = 40 pcs. Since $w_{23} > 0$, go to Step 7.

Step 7: Node 1 is branched into two nodes, viz, node 2 and 3.

At node 2: Set n = n+1 = 2, N1 = {1} + {2} - {1} = {1}.

Set R = R- $\{J_{23}\} = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, S = ϕ, s = 3000 mins, D = 0 and $\underline{Z} = 0$.

Since $R \neq \phi$, find Y_{ij} for all $J_{ij} \varepsilon R$. Find $\max_{J_{ij}\varepsilon R} Y_{ij}$.

It is observed that the maximum value of Y_{ij} is equal to 1.358 Rs./min. corresponding to a job J_{12}. Therefore, $\overline{Z} = \underline{Z} + B_{12}\ [s/\{(S_1X_1 + s_{12})/l_{12} + p_{12}^{(t)}\}] =$ Rs. 6344.00

At node 3: Set $n = n+1 = 3$, $N1 = \{2\} + \{3\} = \{2,3\}$.

For node 3, set $R = R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = S$, $s = d-\tilde{D} = 2676.94$ mins. $D = \tilde{D} = 323.56$ mins. and $\underline{Z} = 40 \times 18 =$ Rs. 720.00, $X_2 = 0$.

Since $R \neq \phi$, calculate Y_{ij} for all jobs $J_{ij} \varepsilon R$.

$\max_{J_{ij}\varepsilon R} Y_{ij}$ corresponds to job J_{12} for which

$Y_{12} = B_{12}/\{(S_1X_1 + s_{12})/l_{12} + p_{12}^{(t)})\} = 1.358$ Rs./min.

Therefore, $\overline{Z} = \underline{Z} + B_{12}\ [s/\{(S_1\ X_1 + s_{12})/l_{12} + p_{12}^{(t)}\}]$

$=$ Rs. 6375.00. Now return to Step 4.

Step 4 finds node 3 with maximum upper bound value. Further computations are done according to the steps 4 through 12 of the algorithm till we get the optimal machining speeds. The optimal solution is listed in Table 2.9.

Table 2.9: Optimal Solution of Numerical Example 4.

Index of Group	Index of Part	Quantity to be produced	Profit Rs.	Optimal machining speed m/min.	Optimal unit production time min/pc
1	1	60	600.00	175.3172	6.787
	2	50	650.00	155.3731	5.521
2	3	40	720.00	127.4141	7.121
3	1	19	123.50	117.7705	8.034
	2	90	765.00	135.5620	10.034
4	1	40	444.00	182.2416	6.416
	2	50	700.00	159.3903	11.225

Total time utilized	=	2,999.66 min.
Total profit	=	Rs. 3,998.50
Total Cost of Production	=	Rs.21,100.00

2.5 HEURISTIC SOLUTION PROCEDURES:

In Sec. 2.4 we presented algorithms for single stage product mix and machine loading problems considering either the maximization of total volume of production or maximization of total profit. For each of these objectives two variations, namely, unit production time for each job is prespecified and the machining speeds are decision variables, were considered. A solution methodology for each one of these situations was developed. The solution methodology utilized the branch and bound philosophy for generating the optimal solution when the total production time required to produce all the jobs to their desired lot sizes turned out to be greater than the available production time on the machine. In order to reduce the computational effort involved in the use of branch and bound procedure, it was felt that some heuristic procedure needs to be developed for identifying the jobs which have every high potential for inclusion in the optimal product mix obtained by using the algorithms given in Sec. 2.4. The following heuristic procedure was developed to identify the set of jobs having high potential for inclusion in the final solution.

As soon as the first step of any of the algorithms given in Section 2.4 reveals that the total available time on the machine is less than the total time required to produce all the jobs to their lot sizes, contribution of each job towards primary objective function, e.g., maximization of total amount of production or the total profit is calculated. The job with the highest contribution is identified and selected for inclusion is the list of selected parts provided the time required to produce the job to its lot size is less than or equal to the available production-time. After the first lot is selected, the lot with the next highest contribution is identified and the procedure is repeated till no job can be selected for inclusion in this list of selected jobs.

If the number of selected jobs in this list turns out to be less than or equal to two, this list is made empty and the branch and bound procedure, i.e. step 3 onwards of the various algorithms are followed as such. However, if the list of selected jobs contains more than two jobs, the last two jobs are removed from the list. The Step 3 of the branch and bound procedure is suitably modified for each of the algorithms given in Section 2.4 to incorporate the heuristic procedure into the branch and bound approach. The remaining steps of the branch and bound procedures, i.e., Step 4 onwards remain the same as in the original algorithms. The modified Step 3 for each case is presented below.

2.5.1 Case 1: Maximization of Total Units Produced:

2.5.1.1 Case 1(a): Prespecified Unit Production Time

Heuristic Algorithm:

The Step 3 of the original algorithm given in Sec. 2.4.1.1 is modified as follows:

Step 3.1. Select an initial node: $n = 1$, $N1 = \{n\}$. For this node, set $R = U$, $S = \emptyset$, $D = 0$, $s = d$, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$. Go to Step 3.2.

Step 3.2: Find from jobs $J_{ij} \in R$, a job $J_{\bar{i}\bar{j}}$ for which $\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}$ is minimum. Find $\bar{D} = D + S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}} + w_{\bar{i}\bar{j}} \, p_{\bar{i}\bar{j}}$, where $w_{\bar{i}\bar{j}}$ max w_{ij}, w_{ij} is varied from zero to $l_{\bar{i}\bar{j}}$ till $\bar{D} \leq d$. Go to Step 3.3.

Step 3.3: If $w_{\bar{i}\bar{j}} = l_{\bar{i}\bar{j}}$ and $\bar{D} = d$, set $S = S + \{J_{\bar{i}\bar{j}}\}$, $\underline{Z} = \sum_{J_{ij} \in S} l_{ij}$ and go to Step 8. If $w_{\bar{i}\bar{j}} = 0$ or $w_{\bar{i}\bar{j}} < l_{\bar{i}\bar{j}}$, go to Step 3.4, otherwise, set $R = R - \{J_{\bar{i}\bar{j}}\}$, $S = S + \{J_{\bar{i}\bar{j}}\}$, $X_{\bar{i}} = 0$, $D = \bar{D}$, $s = d-D$ and go to Step 3.2.

Step 3.4: If $S = \emptyset$ and $w_{\bar{i}\bar{j}} = 0$, then set $Z^* = 0$, $S^* = \emptyset$ and stop. If set S contains more than two jobs, remove the last two jobs selected from the list S and include these jobs in R, otherwise, make S empty. Set $X_i = 1$ for those groups G_i of which no job is included in S. Set $\underline{Z} = \sum_{J_{ij} \in S} l_{ij}$, $D = \{ \sum_{J_{ij} \in S} (s_{ij} + w_{ij} \, p_{ij}) + \sum_{i=1}^{N_g} S_i X_i \}$ and $s = d-D$. Find a job

$J_{\bar{i}\bar{j}}$ from jobs $J_{ij} \varepsilon R$ for which $\{(S_iX_i + s_{ij})/l_{ij}+p_{ij}\}$ is minimum. Set $\bar{Z} = \underline{Z} + [s/\{(S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$ and go to Step 4.

2.5.1.2 Case 1(b): Machining Speeds are Decision Variables:

Heuristic Algorithm:

Step 3 remains the same as given in Sec. 2.5.1.1 except the fact that instead of p_{ij}, $p_{ij}^{(t)}$ given by (2.18) is used.

2.5.2 Case 2: Maximization of Total Profit:

2.5.2.1 Case 2(a): Prespecified Unit Production Time:

Heuristic Algorithm:

The Step 3 of the optimizing algorithm given in Sec. 2.4.2.1 is modified as follows:

Step 3.1: Select initial list, set $n = 1$, $N1 = \{n\}$. For node 1, set $R = U$, $S = \emptyset$, $D = 0$, $s = d$, $\underline{Z} = 0$ and $X_i = 1 \ \forall \ i$. Go to Step 3.2.

Step 3.2: Find a job $J_{\bar{i}\bar{j}}$ from $J_{ij} \varepsilon R$ for which $B_{\bar{i}\bar{j}}/\{(S_{\bar{i}}X_{\bar{i}}+s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}$ is maximum. Calculate $\bar{D} = D + S_{\bar{i}}X_{\bar{i}} + s_{\bar{i}\bar{j}} + w_{\bar{i}\bar{j}}\ p_{\bar{i}\bar{j}}$, where $w_{\bar{i}\bar{j}} = \max w_{ij}$, w_{ij} is varied from zero to $l_{\bar{i}\bar{j}}$ till $\bar{D} \leq d$. Go to Step 3.3.

Step 3.3: If $w_{\bar{i}\bar{j}} = l_{\bar{i}\bar{j}}$ and $\bar{D} = d$, set $S = S + \{J_{\bar{i}\bar{j}}\}$ $\underline{Z} = \sum_{J_{ij}\varepsilon S} B_{ij}\ l_{ij}$ and go to Step 8. If $w_{\bar{i}\bar{j}} = 0$ or $w_{\bar{i}\bar{j}} < l_{\bar{i}\bar{j}}$, go to Step 3.4; otherwise, set $R = R-\{J_{\bar{i}\bar{j}}\}$,

$S = S + \{J_{\bar{i}\bar{j}}\}$, $X_{\bar{i}} = 0$, $D = \bar{\bar{D}}$, $s = d - D$ and go to Step 3.2.

Step 3.4. If $S = \emptyset$ and $w_{\bar{i}\bar{j}} = 0$, set $Z^* = 0$, $S^* = \emptyset$ and stop. If the set S contains more than two job, remove the last two jobs put in list S and include them in R; otherwise, set the list $S = \emptyset$. $X_i = 1$ for those group G_i of which no job is included in S.
Set $\underline{Z} = \sum_{J_{ij} \varepsilon S} w_{ij} B_{ij}$, $D = \{ \sum_{J_{ij} \varepsilon S} (s_{ij} + w_{ij}\, p_{ij}) + \sum_{i=1}^{N_g} S_i X_i \}$ and $s = d - D$. Find a job $J_{\bar{i}\bar{j}}$ from the jobs $J_{ij} \;\varepsilon\; R$ for which $B_{\bar{i}\bar{j}} / \{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}$ is maximum. Set $\bar{\bar{Z}} = \underline{Z} + B_{\bar{i}\bar{j}}\, [s/\{(S_{\bar{i}} X_{\bar{i}} + s_{\bar{i}\bar{j}})/l_{\bar{i}\bar{j}} + p_{\bar{i}\bar{j}}\}]$ and go to Step 4.

2.5.2.1 Case 2(b): Machining speeds are decision variables.

Heuristic Algorithm:

Step 3 remains the same as that given in Sec. 2.5.2.1 except the fact that instead of p_{ij}, $p_{ij}^{(t)}$ given by (2.18) is used.

2.5.3 Computational Experience:

To check suitability and computational efficiency of the heuristic procedure, numerical examples given in Sec. 2.4. were solved using heuristic algorithms. It was observed that in all the cases, the solutions obtained by heuristic procedures and the optimizing solution procedures were the same.

Table 2.10 gives the CPU time taken on DEC-1090 computer system for the various examples when solved using the optimizing solution procedures and the heuristic approaches. It is observed that for all the examples considered the CPU time requirement was lower for the heuristic approach as compared to the optimizing procedure used for solving the example.

Table 2.10: CPU Time Taken by DEC-1090 Computer System.

Problem	CPU time taken by (sec.)	
	Optimizing Solution Procedure	Heuristic Procedure
Numerical Example 1	0.13	0.10
Numerical Example 2	0.16	0.14
Numerical Example 3	0.13	0.11
Numerical Example 4	0.20	0.16

2.5.4 An Illustration Demonstrating Use of Heuristic Procedure:

Numerical example 1 presented in Sec. 2.4.1.1 is considered to demonstrate the heuristic solution procedure.

Solution:

The following is the step-wise solution of the example using the heuristic approach.

Step 1: $Q = 3105$ mins. and $d = 2400$ mins (given).

Since $Q > d$, go to Step 3.

Step 3.1: Select an initial node: set $n = 1$, $N1 = \{n\} = \{1\}$.

For node 1, set $R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \phi$, $D = 0$, $s = 2400$ mins, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$, go to Step 3.2.

Step 3.2: Calculate $\min\limits_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}\}$. Its value is 6.55 min/pc corresponding to job J_{32}. We find that $\bar{\bar{D}} = 393.00$ mins for $w_{32} = 60$ pcs. Go to Step 3.3.

Step 3.3: Since $\bar{\bar{D}} < d$ and $w_{32} = l_{32}$, set $R = R - \{J_{32}\}$

$= \{J_{11}, J_{12}, J_{21}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$,

$S = S + \{J_{32}\} = \{J_{32}\}$, $X_3 = 0$ and $D = \bar{\bar{D}} = 393.00$ mins.

Go to Step 3.2.

Now we recalculate $\{(S_i X_i + s_{ij})/l_{ij} + p_{ij}\}$ for all $J_{ij} \; \varepsilon \; R$. It is observed that the minimum value of the expression is 6.8 min/pc corresponding to job J_{12}.

We find that $w_{12} = 10$ pcs and $\bar{\bar{D}} = 461.00$ mins.

Go to Step 3.3.

Since $w_{12} = l_{12}$ and $\bar{\bar{D}} < d$, set $R = \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{12}, J_{32}\}$, $X_1 = 0$ and $\bar{\bar{D}} = 461.00$ mins.

Similarly following Step 3.2 and 3.3 iteratively, we find that, $R = \{J_{21}, J_{41}, J_{42}\}$, $S = \{J_{11}, J_{12}, J_{22}, J_{23}, J_{31}, J_{32}, J_{43}\}$, $X_i = 0 \; \forall \; i$, $D = 2289.00$ mins. and $w_{42} = 10$. Since $w_{42} < l_{42}$, go to Step 3.4.

$U = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$

$R = U$, $S = \phi$, $s = 2400$ min, $\underline{Z} = 0$, $J_{\bar{i}\bar{j}} = J_{32}$ and $w_{32} = l_{32}$

$R = U - \{J_{32}\}$, $S = \{J_{32}\}$, $s = 2007$ min, $J_{\bar{i}\bar{j}} = J_{12}$ and $w_{12} = l_{12}$

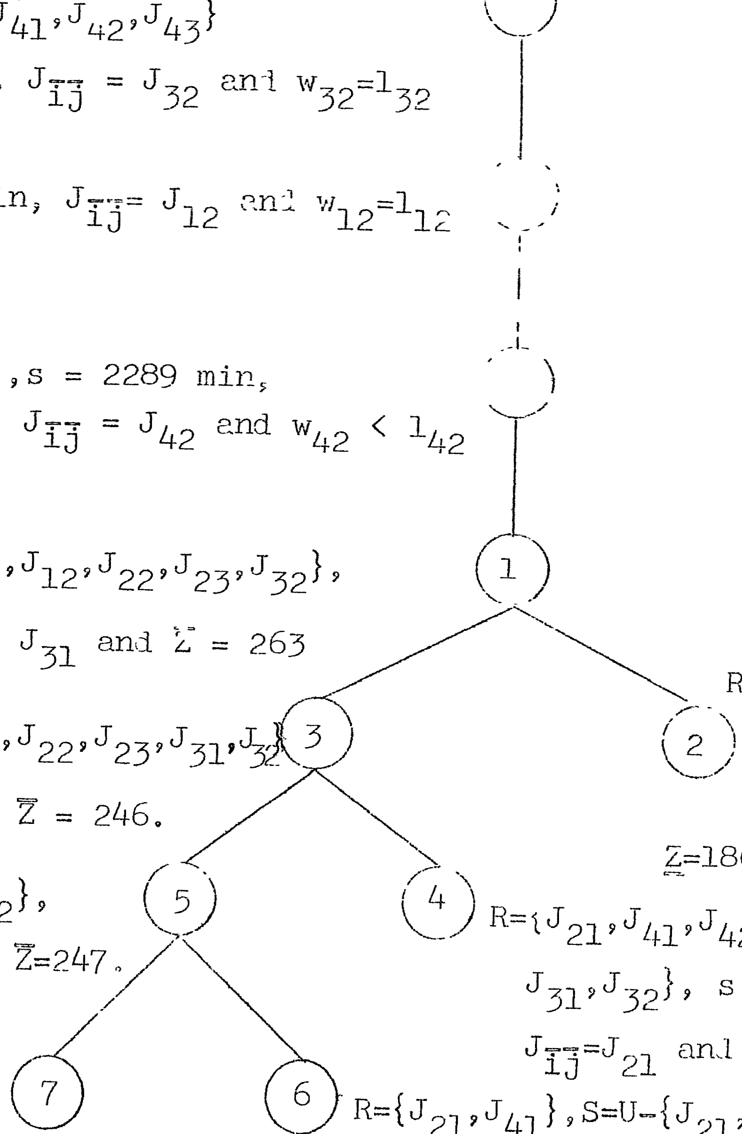
$R = \{J_{21}, J_{41}, J_{42}\}$, $S = U - \{J_{21}, J_{41}, J_{42}\}$, $s = 2289$ min, $J_{\bar{i}\bar{j}} = J_{42}$ and $w_{42} < l_{42}$

$R = \{J_{21}, J_{31}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{11}, J_{12}, J_{22}, J_{23}, J_{32}\}$, $s = 1021$ mins, $\underline{Z} = 180$, $J_{\bar{i}\bar{j}} = J_{31}$ and $\bar{Z} = 263$

$R = \{J_{21}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{11}, J_{12}, J_{22}, J_{23}, J_{31}, J_{32}\}$, $s = 776$ min, $\underline{Z} = 200$, $J_{\bar{i}\bar{j}} = J_{43}$ and $\bar{Z} = 246$.

$R = \{J_{21}, J_{41}, J_{42}\}$, $S = U - \{J_{21}, J_{41}, J_{42}\}$, $s = 111$ min, $\underline{Z} = 240$, $J_{\bar{i}\bar{j}} = J_{42}$ and $\bar{Z} = 247$.

$R = \{J_{21}, J_{41}\}$, $S = U - \{J_{21}, J_{41}\}$, $s = 10$ min, $\underline{Z} = 247$, $J_{\bar{i}\bar{j}} = J_{41}$ and $\bar{Z} = 247$



$R = \{J_{21}, J_{41}, J_{42}, J_{43}$ ... $S = \{J_{11}, J_{12}, J_{22}, J_{2}$ ... $J_{32}\}$, $s = 1021$ m... $\underline{Z} = 180$, $J_{\bar{i}\bar{j}} = J_{43}$ and $\bar{Z} = 24$...

$R = \{J_{21}, J_{41}, J_{42}\}$, $S = \{J_{11}, J_{22}, J_{23}$, $J_{31}, J_{32}\}$, $s = 776$ min, $\underline{Z} = 200$ ... $J_{\bar{i}\bar{j}} = J_{21}$ and $\bar{Z} = 244$.

$R = \{J_{21}, J_{41}\}$, $S = U - \{J_{21}, J_{41}, J_{42}\}$, $s = 111$ mi... $\underline{Z} = 240$, $J_{\bar{i}\bar{j}} = J_{41}$ and $\bar{Z} = 246$.

Fig. 2.3: Tree-Diagram obtained for Numerical Example 1 using Heuristic Approach.

Step 3.4: We remove the last two jobs J_{31} and J_{43} from S and include them in R. Now set $R = \{J_{22}, J_{31}, J_{41}, J_{42}, J_{43}\}$, $S = \{J_{11}, J_{12}, J_{22}, J_{31}, J_{32}\}$, $X_4 = 1$, $\underline{Z} = 180$, $D = 1379.00$ mins. and $s = 1021.00$ mins.

Find $\min\limits_{J_{ij} \varepsilon R} \{(S_i X_i + s_{ij})/l_{ij} + p_{ij}\}$. Its value is 12.25 min/pc corresponding to job J_{31}. Set $\bar{Z} = \underline{Z} + [\ s/\{(S_3 X_3 + s_{31})/l_{31} + p_{31}\}] = 263$ and go to Step 4.

Now we follow branch and bound procedure iterating steps 4 through 7 as discussed in Sec. 2.4.1.1. The final product mix obtained is as shown in Table 2.2. For the purpose of illustration a portion the tree-diagram obtained by the use of heuristic algorithm is depicted in Fig. 2.3.

CHAPTER III

MULTISTAGE PROBLEM

3.1 STATEMENT OF THE PROBLEM:

Consider a multi-stage production system with Ns production stages. Each production stage has limited available processing time. N types of jobs are available for production. Based on the part-family concept of group technology, the jobs have been classified into N_g groups. The i-th group, G_i ($i = 1,2,\ldots, N_g$) has N_i jobs. Each job J_{ij} ($j = 1,2,\ldots, N_i$; $i = 1,2,\ldots, N_g$) has a lot size associated with it.

The objective is to optimally select the jobs and determine the quantities of each to be produced such that the desired criteria of optimization is satisfied subject to the restriction that the total processing time available on each stage is not exceeded. The optimization criteria is either to maximize the number of units produced or maximize the total profit.

The unit processing time required to process a single unit of the job at a given stage is either prespecified or determined based on optimal selection of machining speed. The optimal selection of machining speed is based on

minimum cost of production.

3.2 ASSUMPTIONS:

The various assumptions are:

i) The jobs to be processed on a multi-stage production system do not have any prespecified sequence of operations. Thus, job operations do not have any technological ordering.

ii) The various jobs do not necessarily require processing at all stages of the production system.

iii) Jobs do not have to be necessarily produced to their lot sizes.

iv) Once a job is undertaken for production on a stage, all the units of this job are completed first before taking up the next job.

v) Each unit of the job must be processed to completion.

vi) Each production stage has only one machine.

vii) Only one unit of the job can be processed on the machine at a time.

viii) A limited amount of processing time is available on each machine (each stage). The processing time is the actual productive time available on the machine excluding the time lost due to break-downs, etc.

3.3 NOMENCLATURE:

Following are the notations used for the development of the mathematical models:

a_{ijk} = preparation time for job J_{ij} at k-th stage (min/pc)

b_{ijk} = tool replacement time for job J_{ij} at k-th stage (min/pc).

B_{ij} = profit per unit of job J_{ij} (Rs./pc).

C_{ijk} = 1-minute tool life machining speed for job J_{ij} at k-th stage (m/min)

Cu_k = total utilized processing time of the machine at k-th stage (min)

d_k = available processing time at k-th stage (min)

E_{ijk} = high-efficiency machining speed range for job J_{ij} at k-th stage (m/min)

G_i = i-th group or part-family

i = group index ($i = 1,2,\ldots, N_g$)

j = job index ($j = 1,2,\ldots, N_i$)

J_{ij} = j-th job in i-th group

k = index of the production stage ($k = 1,2,\ldots, Ns$)

l_{ij} = lot size of job J_{ij} (pcs)

n_{ijk} = slope constant of the Taylor tool-life equation for k-th production stage

N = total number of jobs available for production

N_g = total number of part-families or groups

N_i = total number of jobs in group G_i

Ns = total number of stages in the production system

p_{ijk} = unit processing time for J_{ij} at k-th stage (min/pc)

$p_{ijk}^{(c)}$ = minimum-production-cost unit processing time for J_{ij} at k-th stage (min/pc)

$p_{ij}^{(t)}$ = maximum-production-rate unit processing time for J_{ij} at k-th stage (min/pc)

$p_{ij}(v_{ij})$ = machining speed dependent unit processing time for J_{ij} at k-th stage (min/pc)

s_{ijk} = job set-up time at k-th stage for J_{ij} (min)

S_{ik} = group set-up time at k-th stage for group G_i (min)

t_{ijk} = actual machining time at k-th stage for J_{ij} (min/pc)

T_{ijk} = tool life at k-th stage for J_{ij} (min/edge)

U = set of all parts available for production $\{J_{ij}/\ j = 1,2,\ldots,\ N_i;\ i = 1,2,\ldots,\ N_g\}$

v_{ijk} = machining speed for J_{ij} at k-th stage (m/min)

$v_{ijk}^{(c)}$ = minimum-production-cost machining speed for J_{ij} at k-th stage (m/min)

$v_{ijk}^{(e)}$ = high efficiency machining speed for J_{ij} at k-th stage (m/min)

$v_{ijk}^{(t)}$ = maximum-production-rate machining speed used at k-th stage for J_{ij} (m/min)

w_{ij} = quantity of J_{ij} to be produced (pcs)

x_{ij} = 0-1 type variables for J_{ij}

X_i = 0-1 type variable for G_i

Y = total cost of production (Rs.)

S = set of selected jobs at any node

$\sim$ = shows a candidate for acceptance or rejection

$[A]$ = is a Gaussian notation implying a greatest integer less than or equal to A.

3.4 MATHEMATICAL MODELS AND SOLUTION PROCEDURES:

Mathematical models for the multistage problem stated in Sec. 3.1 are presented along with optimal solution methodologies. The maximization of profit is considered as the criteria for optimization. Separate models and solution methodologies are presented for the following two cases:

i) The unit processing time is prespecified.

ii) The machining speed which ultimately determines the unit processing time is a decision variable.

3.4.1 Total Profit Maximization When Unit Processing Time is Prespecified:

3.4.1.1 Constraints:

The following constraints are considered:

i) Machine utilization constraint:

This constraint ensures that the total time needed to process the various jobs at any stage does not exceed the available capacity of that production stage. Mathematically, it can be written as:

$$\sum_{i=1}^{N_g} (S_{ik} X_i + \sum_{j=1}^{N_i} P_{ijk} x_{ij}) \leq d_k \qquad (3.1)$$

$$k = 1, 2, \ldots, Ns$$

where,

P_{ijk} is the job processing time for processing w_{ij} units of job J_{ij} at the k-th stage and is given by,

$$P_{ijk} = s_{ijk} + w_{ij}\, p_{ijk} \,, \tag{3.2}$$

$$x_{ij} = \begin{cases} 1 \text{ if job } J_{ij} \text{ is selected for production} \\ 0 \text{ if job } J_{ij} \text{ is not produced} \end{cases} \tag{3.3}$$

and,

$$X_i = \begin{cases} 1 & \text{if } \sum_{j=1}^{N_i} x_{ij} \geq 1 \\ 0 & \text{if } \sum_{j=1}^{N_i} x_{ij} = 0 \end{cases} \tag{3.4}$$

ii) Lot Size Constraint:

The quantity w_{ij} of job J_{ij} produced should not exceed the lot size specified for it. Thus,

$$0 \leq w_{ij} \leq l_{ij} \qquad \forall \; J_{ij} \in U \tag{3.5}$$

3.4.1.2 Objective function:

The total profit can be expressed as:

$$Z = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} B_{ij}\, w_{ij}\, x_{ij} \tag{3.6}$$

where, B_{ij} is the profit per unit of job J_{ij}, x_{ij} is a 0-1 variable defined by (3.3). The objective is to maximize the total profit, i.e., Z.

3.4.1.3 Solution procedure:

Let Q_k represents the time needed to process all the jobs to their lot sizes at the k-th stage of production. Then,

$$Q_k = \sum_{i=1}^{N_g} \{ S_{ik} + \sum_{j=1}^{N_i} (s_{ijk} + l_{ij}\, p_{ijk})\}$$

$$k = 1,2,\ldots, Ns \qquad (3.7)$$

If for all the production stages the condition $Q^k \leq d^k$ is satisfied, then all the jobs will be produced to their lot sizes. Thus, the optimal solution will be:

$$x^*_{ij} = 1 \qquad \forall\ J_{ij} \in U \qquad (3.8)$$

$$w^*_{ij} = l_{ij} \qquad \forall\ J_{ij} \in U \qquad (3.9)$$

$$Z^* = Z_{max} \qquad (3.10)$$

where,

$$Z_{max} = \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} l_{ij}$$

If the condition $Q^k \leq d^k$ is not satisfied for any of the production stages, then any one of the following situations may arise:

i) All the jobs except one of them, are produced to their lot sizes.

ii) All the jobs are not selected for production. In this situation two cases are possible:

a) Jobs selected for production are produced to their lot size.

b) One of the selected jobs can not be produced to its lot size.

In all the above situations, it is obvious that $Z^* < Z_{max}$. A branch and bound (B & B) procedure is developed to determine its optimal product mix and machine loading for each of these cases.

The B & B procedure is used to branch a node $\tilde{n}$ for which the upper bound value $\bar{Z}$, is maximum. For each of the descendent obtained after branching a node, the upper bound value $\bar{Z}$ and the lower bound value $\underline{Z}$ is determined. To obtain $\bar{Z}$, a job from the remaining jobs is identified for which $B_{ij}\ N_{ij}$ is maximum. The corresponding job is denoted as $J_{\tilde{i}\tilde{j}}$. Mathematically, N_{ij} is expressed as:

$$N_{ij} = \min_{k} [\ s_k/(\frac{S_{ik} + s_{ijk}}{l_{ij}} + p_{ijk})] \qquad (3.11)$$

where s_k is the time available for production at that node. The job $J_{\tilde{i}\tilde{j}}$ is included in the list of selected jobs in one of the two nodes obtained from branching of a node having maximum upper bound value. For the descendent, for which the job $J_{\tilde{i}\tilde{j}}$ is included in the list of selected jobs, the lower bound value $\underline{Z}$, which at the beginning of B & B procedure was set equal to zero, is incremented by $B_{\tilde{i}\tilde{j}}\ w_{\tilde{i}\tilde{j}}$. The list of the selected parts corresponding to the node for which the upper bound value is maximum among all the nodes and equal to the lower bound value at the same node, gives the optimal product mix. At this point the B & B procedure is fathomed.

3.4.1.4 Optimizing algorithm:

The various steps of the algorithm are:

Step 1: Find Q_k, for each stage from (3.7). If $Q_k \leq d_k \ \forall k$, go to Step 2; otherwise, go to Step 3.

Step 2: Produce all the jobs to their lot sizes and terminate the procedure. The optimal solution is given by the following relationships:

$x_{ij}^* = 1$, $w_{ij}^* = l_{ij} \ \forall J_{ij} \in U$ and $Z^* = Z_{max}$.

Step 3: Follow branch and bound procedure:

Select an initial node; set $n = 1$, $N1 = \{n\}$.

For this node, set $R = U$, $S = \phi$, $s_k = d_k$ and $D_k = 0 \ \forall k$, $\underline{Z} = 0$ and $X_i = 1 \ \forall i$. Find,

$$\text{Find} \quad N_{ij} = \min_k \left[\frac{s_k}{\frac{S_{ik} X_i + s_{ijk}}{l_{ij}} + p_{ijk}} \right] \ \forall J_{ij} \in R.$$

Find $\max\limits_{J_{ij} \in R} B_{ij} N_{ij}$ and denote the corresponding job by $J_{\bar{i}\bar{j}}$. Set, $\overline{Z} = B_{\bar{i}\bar{j}} N_{\bar{i}\bar{j}}$ and go to Step 4.

Step 4: Find a node from the nodes $n \in N1$ for which $\overline{Z}$ is maximum. Denote this node by $\tilde{n}$ and $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. For the node $\tilde{n}$, if $\overline{Z} = \underline{Z}$ go to Step 8; otherwise, go to Step 5.

Step 5: Find for each stage $\tilde{D}_k = D_k + S_{\tilde{i}k} X_{\tilde{i}} + s_{\tilde{i}\tilde{j}k} + w_{\tilde{i}\tilde{j}} p_{\tilde{i}\tilde{j}k}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$, w_{ij} is varied from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D}_k \leq d_k \ \forall k$. If $w_{\tilde{i}\tilde{j}} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: Remove the job $J_{\tilde{i}\tilde{j}}$ from the set R of the node $\tilde{n}$. Set $R = R - \{J_{\tilde{i}\tilde{j}}\}$. If $R = \phi$, set $\overline{Z} = \underline{Z}$; otherwise, find, $\max\limits_{J_{ij} \varepsilon R} B_{ij}\ N_{ij}$, denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\ N_{\bar{i}\bar{j}}$. Return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes, as given below:

At first Node:

Set $n = n+1$, $N1 = N1 + \{n\} - \{\tilde{n}\}$.

For the newly cre ted node n, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$, $S = \phi$, $s_k = s_k$ and $D_k = D_k \ \forall\ k$, $N_{ij} = N_{ij} \ \forall\ J_{ij}\ \varepsilon\ R$ and $\underline{Z} = \underline{Z}$. If $R = \phi$, set $\overline{Z} = \underline{Z}$; otherwise, find $\max\limits_{J_{ij} \varepsilon R} B_{ij}\ N_{ij}$, denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\ N_{\bar{i}\bar{j}}$.

At second Node:

Set $n = n+1$ and $N1 = N1 + \{n\}$.

For the newly created node n, set $R = R$, $S = S + \{J_{\tilde{i}\tilde{j}}\}$, $D_k = \tilde{D}_k$ and $s_k = d_k - \tilde{D}_k \ \forall\ k$, $X_i = 0$ and $\underline{Z} = \underline{Z} + B_{\tilde{i}\tilde{j}}\ w_{\tilde{i}\tilde{j}}$. If $R = \phi$, set $\overline{Z} = \underline{Z}$; otherwise, recalculate the value of $N_{ij} \ \forall\ J_{ij}\ \varepsilon\ R$ and find $\max\limits_{J_{ij} \varepsilon R} B_{ij}\ N_{ij}$. Denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\overline{Z} = \underline{Z} + B_{\bar{i}\bar{j}}\ N_{\bar{i}\bar{j}}$.

Return to Step 4.

Step 8: The optimal solution is contained in node $\tilde{n}$. Set, $S^* = S$, $Z^* = \underline{Z}$ and stop.

The value of $w_{\tilde{i}\tilde{j}}$ required in Step 5 is determined using the following relationship.

$$w_{\tilde{i}\tilde{j}} = \min_{k} \left[\frac{d_k - D_k - S_{\tilde{i}k} X_i - s_{\tilde{i}\tilde{j}k}}{p_{\tilde{i}\tilde{j}k}} \right]$$

If, $w_{\tilde{i}\tilde{j}} > l_{\tilde{i}\tilde{j}}$, we set $w_{\tilde{i}\tilde{j}} = l_{\tilde{i}\tilde{j}}$.

3.4.1.5 Numerical Example 5:

Consider a two stage production system for which ten types of jobs are available for production. These jobs are classified into four groups. The basic data for the problem is given in Table 3.1. The objective is to determine optimal machine loading and product mix.

3.4.1.6 Solution:

Using the various steps of the algorithm, we obtain,

Step 1: Q_1 = 3105.00 mins. and Q_2 = 2995.00 mins.

Since $Q_k > d_k \; \forall \; k$, go to Step 3.

Step 2: Select an initial node: Set $n = 1$, $N1 = \{n\} = \{1\}$.

For node 1, set $R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \phi$, s_1 = 2400.00 mins, s_2 = 2400.00 mins, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$. Calculate $N_{ij} \; \forall \; J_{ij} \in R$. Find $\max_{J_{ij} \in R} B_{ij} N_{ij}$. Its value is Rs. 4080.00 corresponding to job J_{12}. Therefore, $\bar{Z} = B_{12} N_{12}$ = Rs. 4080.00.

Go to Step 4.

Step 4: Since for node 1, $\bar{Z} \neq \underline{Z}$, go to Step 5.

Step 5: For w_{12} = 10 pcs, $\tilde{D}_1$ = 68.00 mins and $\tilde{D}_2$ = 94.00 mins.

Since $\tilde{D}_k < d_k \; \forall \; k$, go to Step 7.

Step 7: Branch node 1 into two nodes, viz, nodes 2 and 3.

At node 2: Set $n = n+1 = 2$, $N1 = \{1\} + \{2\} - \{1\} = \{2\}$.

$R = \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$

Table 3.1: Basic Data for Numerical Example 5.

Group	Job	Profit per Unit (Rs./pc)	Lot Size (pcs)	Group Set-up Time (mins.)		Job Set-up Time (min/lot)		Unit Production Time (min/pc)	
i	j	B_{ij}	l_{ij}	S_{i1}	S_{i2}	s_{ij1}	s_{ij2}	p_{ij1}	p_{ij2}
1	1	6.50	30	40	45	19	8	6	4
	2	16.00	10			8	19	2	3
2	1	19.50	20	35	20	10	15	17	8
	2	12.50	50			9	9	9	7
	3	9.00	30			15	10	7	9
3	1	7.50	20	20	35	5	[illegible]	12	2
	2	14.00	60			13	5	6	12
4	1	9.00	20	45	40	6	20	16	19
	2	12.00	10			10	6	13	15
	3	14.80	40			20	10	15	13

Available Processing Time :

At stage 1 d_1 = 40 hrs.
At stage 2 d_2 = 40 hrs.

Table 3.2: Optimal Solution of Numerical Example 5.

Index of Accepted Group	Index of Accepted Job	Quantity to be produced pcs	Profit obtained Rs.
1	1	30	195.00
	2	10	160.00
2	1	20	390.00
	2	50	625.00
	3	30	270.00
3	2	60	840.00
4	3	40	592.00

Total number of units produced = 240

Total profit obtained = Rs. 3,072.00

Total processing time utilized:

At stage 1 = 2,394.00 mins.

At stage 2 = 2,386.00 mins.

and $S = \phi$. We find that $s_1 = s_2 = 2400.00$ mins, $\underline{Z} = 0$ and $X_i = 0 \; \forall \; i$. Since $R \neq \phi$, find $\max\limits_{J_{ij} \varepsilon R} B_{ij} N_{ij}$. Its value is Rs. 3025.00 corresponding to a job J_{22}. Set $\overline{Z}$ = Rs. 3025.00.

At node 3: $n = n+1 = 3$, $N1 = \{2\} + \{3\} = \{2,3\}$. $R = \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$ and $S = \{J_{12}\}$. We find that $s_1 = 2332.00$ mins, $s_2 = 2306.00$ min, $D_1 = 68.00$ mins., $D_2 = 94.00$ mins, $\underline{Z}$ = Rs. 160.00 and $X_1 = 0$. Recalculate the value of $N_{ij} \; \forall \; J_{ij} \; \varepsilon \; R$. Find $\max\limits_{J_{ij} \varepsilon R} B_{ij} N_{ij}$. Its value corresponds to a job J_{22}. Therefore, $\overline{Z} = \underline{Z} + B_{22} N_{22} =$ = Rs. 3110.0.

Return to Step 4.

From Step 4, we find that $\max\limits_{n \varepsilon N1} \overline{Z}$ occurs at node 3. Therefore, the further branching takes place at this node. The steps 4 to 7 of the algorithm are repeated till the branching procedure is fathomed. The optimal solution obtained is given in Table 3.2.

3.4.2 Total Profit Maximization when Machining Speeds are Decision Variables:

Before developing a mathematical model and solution procedure for the problem, we develop mathematical expressions for job processing time and cost for processing a single unit at a given stage of production.

3.4.2.1 Job processing time:

The unit processing time, $p_{ijk}\ (v_{ijk})$, required to process a single unit of the job J_{ij} at the k-th production stage is a function of machining speed v_{ijk} for that job at k-th production stage. Mathematically,

$$p_{ijk}\ (v_{ijk}) = a_{ijk} + t_{ijk} + b_{ijk}\ \frac{t_{ijk}}{T_{ijk}}$$

or,

$$p_{ijk}\ (v_{ijk}) = a_{ijk} + \frac{\lambda_{ijk}}{v_{ijk}} + \frac{\lambda_{ijk}\ b_{ijk}\ v_{ijk}^{(1/n_{ijk}-1)}}{C_{ijk}^{1/n_{ijk}}}$$

$$\forall\ J_{ij} \in U \qquad (3.12)$$

where, a_{ijk}, b_{ijk} and t_{ijk}, respectively represent the preparation time, tool replacement time, and actual machining time on k-th production stage. T_{ijk}, n_{ijk} and C_{ijk} are the parameters of the Taylor tool life equation.

Let $v_{ijk}^{(t)}$ represent the machining speed at which $p_{ijk}\ (v_{ijk})$ is minimum. $v_{ijk}^{(t)}$ is obtained by setting the first derivative of $p_{ijk}\ (v_{ijk})$ with respect to v_{ijk} equal to zero.

$$v_{ijk}^{(t)} = C_{ijk}/\left(\frac{b_{ijk}}{\sqrt{n_{ijk}-1}}\right)^{n_{ij}} \qquad i = 1,2,\ldots,N_g \qquad (3.13)$$

$$j = 1,2,\ldots,N_i$$

$$k = 1,2,\ldots,Ns$$

If w_{ij} units of job J_{ij} are processed on k-th stage at a speed v_{ijk}, the job processing time, P_{ijk}, is given by,

$$P_{ijk} = s_{ijk} + w_{ij}\ p_{ijk}\ (v_{ijk}) \qquad (3.14)$$

3.4.2.2 Production Cost:

The cost of processing one unit of job J_{ij} at stage k, $q_{ijk}(v_{ijk})$, is also a function of the machining speed v_{ijk} at that stage. Mathematically, it is expressed as:

$$q_{ijk}(v_{ijk}) = \alpha_k a_{ijk} + (\alpha_k + \beta_{ijk}) \frac{\lambda_{ijk}}{v_{ijk}} + (\alpha_k b_{ijk} + \gamma_{ijk}) \frac{\lambda_{ijk} v_{ijk}^{(1/n_{ijk}-1)}}{C_{ijk}^{1/n_{ijk}}} \tag{3.15}$$

where α_k, β_{ijk}, γ_{ijk}, respectively represent the direct labour cost, machining overhead cost and tool cost for job J_{ij} at production stage k.

Let $v_{ijk}^{(c)}$ represent the 'Minimum-Production-Cost Machining Speed'. It is obtained by setting the first derivative of (3.15) with respect to v_{ijk} equal to zero. Mathematically,

$$v_{ijk}^{(c)} = C_{ijk} \left\{ \frac{(\alpha_k + \beta_{ijk})}{(1/n_{ijk} - 1)\ (\alpha_k b_{ijk} + \gamma_{ijk})} \right\}^{n_{ij}} \tag{3.16}$$

3.4.2.3 Objective Function:

Our primary objective is to determine product mix and machining speeds such that the total profit is maximized. However, once the values of optimal decision variables which yield maximum profit are obtained, the machining speeds can be further manipulated such that the cost of production is

minimized without sacrificing the maximum profit. The primary objective function turns out to be the same as given by (3.6). The secondary criteria of minimizing the total cast of production involves the total production cost expression which is given as:

$$Y = \sum_{k=1}^{Ns} \sum_{i=1}^{N_g} \{\alpha_k S_{ik} X_i + \sum_{j=1}^{N_i} (\alpha_k s_{ijk} + w_{ij} q_{ijk} (v_{ijk})) x_{ij}\} \quad (3.17)$$

3.4.2.4 Analysis:

For all the jobs, the minimum-production-time machining speed and unit processing time at the various production stages are calculated from (3.13) and (3.12), respectively. The following expression is used to calculate the job processing time for each job at each stage considering the fact that the each unit in the lot associated with that job is produced at minimum unit processing time.

$$P_{ijk}^{(t)} = s_{ijk} + l_{ij} p_{ijk} (v_{ijk}^{(t)}) \quad (3.18)$$

Mathematically, Q_k^t, the time needed by k-th production stage to process all the jobs to their lot sizes at minimum-production-time machining speeds, can be expressed as,

$$Q_k^t = \sum_{i=1}^{N_g} (S_{ik} + \sum_{j=1}^{N_i} P_{ijk}^{(t)}), \quad k = 1, 2, \ldots, Ns \quad (3.19)$$

Depending upon the value of Q_k^t, there are three possible situations : $Q_k^t = d_k \; \forall k$, $Q_k^t > d_k$ for at least

one of the production stages and $Q_k^t \leq d_k \ \forall k$. These three cases are discussed separately.

i) $Q_k^t = d_k \ \forall k$:

If $Q_k^t = d_k \ \forall k$ ($k = 1,2,\ldots, Ns$), then produce all the jobs to their lot sizes (at the minimum-production-time machining speeds), In this situation, the optimal product mix and the cutting parameters are given by the following relationships:

$$\left.\begin{array}{ll} & x_{ij}^* = 1 \\ & w_{ij}^* = l_{ij} \\ \text{and} & v_{ijk}^* = v_{ijk}^{(t)} \quad k = 1,2,\ldots, Ns \end{array}\right\} \forall J_{ij} \varepsilon U \qquad (3.20)$$

ii) $Q_k^t > d_k$ for at least one of the production stages:

In this case, the processing time available on at least one of the production stages is not sufficient to process all the jobs to their lot sizes. In fact, any one of the following situations may result:

i) All the jobs, except one are produced to their lot sizes.

ii) All the jobs are not produced and

a) the jobs selected are produced to their lot sizes.

b) all these jobs except one are produced to their lot sizes.

In all the above situations, the optimal product mix and machining speeds can be obtained using the following approach.

First of all, we determine the optimal machine loading and product mix based on the minimum-production-time speeds at each stage using the solution methodology discussed in Sec.3.4.1. Let S^* and w^*_{ij} ($J_{ij} \in S^*$) respectively represent the set of jobs accepted for production and the quantities of each job to be produced. Following is the optimal solution.

$$x^*_{ij} = \begin{cases} 1 & \forall\ J_{ij} \in S^* \\ 0 & \text{Otherwise} \end{cases} \tag{3.21}$$

Further,

$$X_i^* = \begin{cases} 1 & \text{If } \sum_{j=1}^{N_i} x_{ij} \geq 1 \\ 0 & \text{Otherwise} \end{cases} \tag{3.22}$$

For each production stage, the total time consumed Cu_k and the slack time δ_k are calculated from the following expressions.

$$Cu_k = \sum_{i=1}^{N_g} \{S_{ik}\, X_i^* + \sum_{j=1}^{N_i} (s_{ijk} + w^*_{ij}\ \ p_{ijk}^{(t)})\ x^*_{ij}\ \} \tag{3.23}$$

and $$\delta_k = d_k - Cu_k \tag{3.24}$$

If $\delta_k = 0$, then all the selected jobs are processed at their minimum-production-time machining speeds (at that stage). Obviously,

$$v^{*\,\prime}_{ijk} = v_{ijk}^{(t)} \quad \forall\ J_{ij} \in U \tag{3.25}$$

If $\delta_k > 0$, it implies that there is some slack time available at the k-th stage. This slack time can be advantageously utilized to minimize the total cost of production. Since $v_{ijk}^{(t)} > v_{ijk}^{(c)}$ and $q_{ijk}\ (v_{ijk}^{(t)}) > q_{ijk}\ (v_{ijk}^{(c)})$, it is

reasonable to assume that the cost of processing a single unit of a job is least at the minimum-production-cost machining speed and it increases with an increase in the machining speed upto the minimum-production-time machining speed. Therefore, the slack time can be utilized in lowering down the cost of production by decreasing the machining speed from $v_{ijk}^{(t)}$ to $v_{ijk}^{(c)}$. Thus, the objective is to determine v_{ijk}^{*}, $v_{ijk}^{(t)} \leq v_{ijk}^{*} < v_{ij}^{(t)}$, such that the cost of production is minimized. The problem is formulated as a nonlinear program as given below:

$$\text{Minimize} \quad Y = \sum_{k=1}^{Ns} \sum_{i=1}^{N_g} \{\alpha_k S_{ik} X_i^{*} + \sum_{j=1}^{N_i} (\alpha_k s_{ijk} + w_{ij}^{*} q_{ijk} (v_{ijk})) x_{ij}^{*} \} \tag{3.26}$$

subject to

$$\sum_{i=1}^{N_g} \{S_{ik} X_i^{*} + \sum_{j=1}^{N_i} (s_{ijk} + w_{ij}^{*} p_{ijk} (v_{ijk})) x_{ij}^{*} \} \leq d_k \qquad k = 1,2,\ldots, Ns \tag{3.27}$$

There are certain constant terms in (3.26) and (3.27) which can be ignored for the purpose of optimization. The problem can be restated as:

$$\text{Minimize} \quad Y' = \sum_{k=1}^{Ns} \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^{*} q_{ijk} (v_{ijk}) x_{ijk}^{*} \tag{3.28}$$

subject to,

$$\sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^{*} p_{ijk} (v_{ijk}) x_{ij}^{*} \leq do_k \quad k = 1,2,\ldots, Ns \tag{3.29}$$

where,

$$Y' = Y - \sum_{k=1}^{Ns} \sum_{i=1}^{N_g} (\alpha_k S_{ik} X_i^* + \sum_{j=1}^{N_i} \alpha_k s_{ijk} x_{ij}^*) \qquad (3.30)$$

and,

$$W_k = d_k - \sum_{i=1}^{N_g} (S_{ik} X_i^* + \sum_{j=1}^{N_i} s_{ijk} x_{ij}^*) \quad k = 1, 2, \ldots, Ns \qquad (3.31)$$

For the problem represented by (3.28) and (3.29), a Lagrangian function of the following form is obtained.

$$L (v_{ijk}, \mu_k) = \sum_{k=1}^{Ns} \sum_{i=1}^{N_g} \sum_{j=1}^{Ns} w_{ij}^* q_{ijk} (v_{ijk}) x_{ijk}^*$$

$$+ \sum_{k=1}^{Ns} \mu_k \{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* p_{ijk} (v_{ijk}) x_{ij}^*$$

$$+ K_k^2 - do \}$$

where μ_k and K_k represent the Lagrangian multiplier and slack variable, respectively. For optimal values of v_{ijk}^*, μ_k^* & K_k^*, we set $\partial Z / \partial v_{ijk} = 0$. It gives

$$\frac{\partial q_{ijk} (v_{ijk}^*)}{\partial v_{ijk}^*} + \frac{\partial p_{ijk} (v_{ijk}^*)}{\partial v_{ijk}^*} = 0$$

or,

$$- \left(\frac{\alpha_k + \beta_{ijk}}{v_{ijk}^2} \right) \lambda_{ijk} + (\alpha_k b_{ijk} + \gamma_{ijk})$$

$$\left(\frac{\lambda_{ijk} v_{ijk}^{*(1/n_{ijk} - 2)}}{C_{ij}^{1/n_{ij}}} \right)$$

$$+ \mu_k^* \left(-\frac{\lambda_{ijk}}{v_{ijk}^*} + \frac{\lambda_{ijk}\, b_{ijk}\, v_{ijk}^{*\,(1/n_{ijk}-2)}}{C_{ijk}^{1/n_{ijk}}} \right) = 0$$

From the above equation, we get

$$\mu_k^* = \frac{\beta_{ijk} - (1/n_{ijk}-1)\, \gamma_{ij}\, (v_{ijk}^* / C_{ijk})^{1/n_{ijk}}}{-1 + (1/n_{ijk}-1)\, b_{ijk}\, (v_{ijk}^* / C_{ijk})^{1/n_{ijk}}} - \alpha_k$$

$$k = 1,2,\ldots, Ns \qquad (3.32)$$

and,

$$v_{ijk}^* = C_{ijk} \left\{ \frac{\alpha_k + \mu_k^* + \beta_{ijk}}{(\alpha_k + \mu_k^* b_{ijk} + \gamma_{ijk})(1/n_{ijk}-1)} \right\}^{n_{ijk}}$$

$$i = 1,2,\ldots, N_g$$
$$j = 1,2,\ldots, N_i$$
$$k = 1,2,\ldots, Ns \qquad (3.33)$$

Similarly, setting $\partial Z/\partial \mu_k = 0$ yields,

$$K_k^{*2} = do_k - \sum_{i=1}^{N_g} \sum_{j=1}^{N_i} w_{ij}^* \, p_{ijk}\, (v_{ijk}^*)\, x_{ij}^* , \quad k = 1,2,\ldots, Ns \qquad (3.34)$$

and $\partial Z/\partial K_k = 0$ gives,

$$\mu_k^* K_k^* = 0 \quad , \; k = 1,2,\ldots, Ns \qquad (3.35)$$

If we substitute $v_{ijk}^* = v_{ijk}^{(c)}$ in (3.32), we find that $\mu_k^* = 0$. Similarly, if we substitute $v_{ijk}^* = v_{ijk}^{(t)}$ in (3.32), we obtain $\mu_k^* = \infty$. Hence, it is obvious that,

$$\mu_k^* \geq 0 \quad \text{for } v_{ijk}^* \in (v_{ijk}^{(c)}, v_{ijk}^{(t)}), \; k = 1,2,\ldots, Ns \qquad (3.36)$$

Since μ_k^* is non-negative, it guarantees the minimization of cost of processing selected jobs on k-th production stage.

If $\mu_k^* = 0$, then from (3.33), $v_{ijk}^* = v_{ijk}^{(c)} \quad \forall J_{ij} \in S^*$.
However, if $\mu_k^* > 0$, then the optimal values of μ_k^* and $v_{ijk}^* \in E_{ijk}$ for each of the production stage is determined from (3.32) and (3.33), respectively, by using hit and trial approach. E_{ijk} is the high-efficiency machining speed range for job J_{ij} at stage k. Mathematically,

$$E_{ijk} = [v_{ijk}^{(c)}, v_{ijk}^{(t)}] \tag{3.37}$$

iii) $\underline{Q_k^t \leq d_k^t \;\forall k}$:

When $Q_k^t \leq d_k^t \;\forall k$, the optimal solution turns out to be the following:

$$R' = \phi, \; S^* = U, \; X_i^* = 1 \;\forall i, \; x_{ij}^* = 1 \text{ and } w_{ij}^* = l_{ij} \;\forall J_{ij} \in S^* \tag{3.38}$$

For the production stages for which $Q_k^t = d_k^t$, the optimal machining speeds v_{ijk}^* are

$$v_{ijk}^* = v_{ijk}^{(c)} \quad \forall J_{ij} \in S^*.$$

However, for the production stages for which $Q_k^t < d_k^t$, the optimal machining are determined as follows:

We calculate the slack time,

$$\delta_k = d_k - Q_k^t$$

The optimal machining speeds are determined by formulating a nonlinear program of the problem as given by (3.26) and (3.27). Hit and trial approach is used to obtain the optimal solution from (3.32) and (3.33).

3.4.2.5 Optimizing Algorithm:

The following is the step by step description of the optimizing algorithm:

Step 1: Find Q_k^t, for each stage from (3.19). If $Q_k^t \le d_k \; \forall \; k$, go to Step 2, otherwise, go to Step 3.

Step 2: The optimal solution is:

$$S^* = U$$

$$x_{ij}^* = 1 \quad \forall \; J_{ij} \in S^*$$

$$w_{ij}^* = l_{ij} \quad \forall \; J_{ij} \in S^*$$

$$X_i^* = 1 \quad \forall \; i$$

$$Z^* = \sum_{J_{ij} \in S} B_{ij} \, w_{ij}^*$$

Set k = 1 and go to Step 9.

Step 3: Follow branch and bound procedure.

Select an initial node: set h = 1, N1 = {n}.

For this node, set R = U, S = $\emptyset$, $s_k = d_k$ and $D_k = 0 \; \forall \; k$, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$.

$$\text{Find } N_{ij} = \min_k \left[\frac{s_k}{\dfrac{S_{ik} X_i + s_{ijk}}{l_{ij}} + p_{ijk}(t)} \right] \forall \; J_{ij} \in R.$$

Find $\max_{J_{ij} \varepsilon R} B_{ij} N_{ij}$ and denote the corresponding job by $J_{\bar{i}\bar{j}}$. Set, $\bar{Z} = B_{\bar{i}\bar{j}} N_{\bar{i}\bar{j}}$ and go to Step 4.

Step 4: Find a node from the nodes $n \varepsilon N1$ for which $\bar{Z}$ is maximum. Denote this node by $\tilde{n}$ and $J_{\bar{i}\bar{j}}$ in $\tilde{n}$ by $J_{\tilde{i}\tilde{j}}$. For the node $\tilde{n}$, if $\bar{Z} = \underline{Z}$, go to Step 8; otherwise, go to Step 5.

Step 5: Find for each stage $\tilde{D}_k = D_k + S_{\tilde{i}k} X_{\tilde{i}} + s_{\tilde{i}\tilde{j}k} + w_{\tilde{i}\tilde{j}} \ p_{\tilde{i}\tilde{j}k}$, where $w_{\tilde{i}\tilde{j}} = \max w_{ij}$, w_{ij} is varied from zero to $l_{\tilde{i}\tilde{j}}$ till $\tilde{D}_k \leq d_k \ \forall k$. If $w_{\tilde{i}\tilde{j}} = 0$, go to Step 6; otherwise, go to Step 7.

Step 6: Remove the job $J_{\tilde{i}\tilde{j}}$ from the set R of the node $\tilde{n}$. Set $R = R - \{J_{\tilde{i}\tilde{j}}\}$. If $R = \phi$, set $\bar{Z} = \underline{Z}$; otherwise, find $\max_{J_{ij} \varepsilon R} B_{ij} N_{ij}$, denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{Z} = \underline{Z} + B_{\bar{i}\bar{j}} N_{\bar{i}\bar{j}}$.
Return to Step 4.

Step 7: Branch the node $\tilde{n}$ into two nodes, as given below:
At first node: Set $n = n+1$, $N1 = N1 + \{n\} - \{\tilde{n}\}$.
For the newly created node n, set $R = R - \{J_{\tilde{i}\tilde{j}}\}$, $S = \phi$, $s_k = s_k$ and $D_k = D_k \ \forall k$, $N_{ij} = N_{ij} \ \forall J_{ij} \varepsilon R$ and $\underline{Z} = \underline{Z}$.
If $R = \phi$, set $\bar{Z} = \underline{Z}$; otherwise, find $\max_{J_{ij} \varepsilon R} B_{ij} N_{ij}$, denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{Z} = \underline{Z} + B_{\bar{i}\bar{j}} N_{\bar{i}\bar{j}}$.

At second node: Set $n = n+1$, $N1 = N1 + \{n\}$.

For the node n, set $R = R$, $S = S+\{J_{\tilde{i}\tilde{j}}\}$, $D_k = \tilde{D}_k$ and $s_k = d_k - D_k \; \forall\, k$, $X_{\tilde{i}} = 0$ and $\underline{Z} = \underline{Z} + B_{\tilde{i}\tilde{j}} N_{\tilde{i}\tilde{j}}$. If $R = \emptyset$, set $\bar{Z} = \underline{Z}$; otherwise, recalculate the value of $N_{ij} \; \forall\, J_{ij} \in R$ and find $\max_{J_{ij} \in R} B_{ij} N_{ij}$. Denote the corresponding job by $J_{\bar{i}\bar{j}}$ and set $\bar{Z} = \underline{Z}\, B_{\bar{i}\bar{j}} N_{\bar{i}\bar{j}}$.

Return to Step 4.

Step 8: The optimal solution is contained in node $\tilde{n}$. Set $S^* = S$, $Z^* = \underline{Z}$, $Q_k = D_k \; \forall\, k$, $k = 1$ and go to Step 9.

Step 9: If $Q_k = d_k$, set $v^*_{ijk} = v_{ijk}^{(t)} \; \forall\, J_{ij} \in S^*$ and go to Step 13; otherwise, if $Q_k < d_k$, find $v_{ijk}^{(c)}$ and $p_{ijk}^{(c)}$ from (3.13) and (3.12) respectively. Calculate,

$$Q_k = \sum_{i=1}^{N_g} \{S_{ik} X^*_i + \sum_{j=1}^{N_i} (s_{ijk} + w^*_{ij}\, p_{ijk}^{(c)} \quad x^*_{ij} \}$$

If $Q_k \leq d_k$, set $v^*_{ijk} = v_{ijk}^{(c)} \; \forall\, J_{ij} \in S^*$ and go to Step 13; otherwise go to Step 10.

Step 10. Select a job $J_{ij} \in S^*$. Find initial value of μ_k from the following expression:

$$\mu_k = \frac{\beta_{ijk} - (1/n_{ijk} - 1)\, \gamma_{ijk} \left(\dfrac{v_{ijk}^{(t)} + v_{ijk}^{(c)}}{2C_{ijk}} \right)^{1/n_{ijk}}}{-1 + (1/n_{ijk} - 1)\, b_{ijk} \left(\dfrac{v_{ijk}^{(t)} + v_{ijk}^{(c)}}{2C_{ijk}} \right)^{1/n_{ijk}}}$$

Go to Step 11.

<u>Step 11</u>: For the value of μ_k obtained in Step 10, find $v_{ijk}^{(e)}$ and p_{ijk} from (3.33) and (3.12), respectively. Calculate,

$$Q_k = \sum_{i=1}^{N_g} \{S_{ik} X_i^* + \sum_{j=1}^{N_i} (s_{ijk} + w_{ij} p_{ijk}) x_{ij}^* \}$$

If $Q_k = d_k$, set $v_{ijk}^* = v_{ijk}^{(e)} \forall J_{ij} \varepsilon S^*$ and go to Step 13; otherwise, go to Step 12.

<u>Step 12</u>: Modify the value of μ_k appropriately depending on whether $Q_k < d_k$ or $Q_k > d_k$. If $Q_k < d_k$, reduce the value of μ_k and if $Q_k > d_k$, increase the value of μ_k.

<u>Step 13</u>: If $k < Ns$, set $k = k+1$ and go to Step 9; otherwise, stop.

3.4.2.6 <u>Numerical Example 6.</u>

Consider a two stage production system for which ten types of jobs are available for production. These jobs are classified into four groups. The objective is to determine optimal machine loading and product mix when the machining speeds are decision variables. The basic data for the example considered is given in Tables 3.3(a) and 3.3(b).

3.4.2.7 <u>Solution:</u>

To find the optimal solution of the problem given in Sec. 3.4.2.6, we follow the various steps of the optimizing algorithm given in Sec. 3.4.2.5. The optimal solution is presented in Table 3.4.

Table 3.3 (a): Basic Data for Numerical Example 6.

Group	Part	Profit Per Unit (Rs./pc)	Lot Size (pcs)	Tool Life Parameters: Slope Constant		Tool Life Parameters: 1-min. tool Life machining Speed (m/min)		Cost Parameters (Rs./min): Machining Overhead		Cost Parameters (Rs./min): Tool Cost	
i	j	B_{ij}	l_{ij}	n_{ij1}	n_{ij2}	C_{ij1}	C_{ij2}	β_{ij1}	β_{ij3}	γ_{ij1}	γ_{ij2}
1	1	10.00	60	0.25	0.25	350	350	10	20	400	550
	2	13.00	50	0.25	0.25	350	250	25	30	750	600
2	1	8.00	100	0.33	0.20	400	350	15	20	600	450
	2	2.00	70	0.25	0.20	250	200	20	25	800	600
	3	18.00	40	0.20	0.33	240	300	15	15	450	500
3	1	6.50	30	0.25	0.20	250	240	30	15	600	450
	2	8.50	90	0.33	0.25	300	250	15	20	500	800
4	1	11.00	40	0.20	0.33	350	400	20	15	450	600
	2	14.00	50	0.25	0.25	350	350	20	10	550	400
	3	3.50	80	0.20	0.25	200	350	25	25	600	750

Table 3.3(b): Basic Data for Numerical Example 6

Group	Part	Machining Constant		Time Parameters: Group Set-up Time (min)		Job Set-up Time (min/lot)		Preparation Time (min/pc)		Tool Replacement Time (min/edge)	
i	j	λ_{ij1}	λ_{ij2}	S_{i1}	S_{i2}	s_{ij1}	s_{ij2}	a_{ij1}	a_{ij2}	b_{ij1}	b_{ij2}
1	1	707	1257	20	23	19.0	10.0	2.5	3.0	2.0	4.0
	2	377	565			8.0	5.0	3.0	3.0	3.5	3.0
2	1	1257	424	22	35	10.0	6.0	3.0	4.0	3.0	3.5
	2	982	524			9.0	20.0	4.0	2.5	2.5	4.5
	3	565	626			15.0	13.0	2.5	5.0	2.5	1.5
3	1	565	565	15	40	5.0	15.0	3.0	2.5	3.0	2.5
	2	626	982			13.0	9.0	5.0	4.0	1.5	2.5
4	1	424	1257	25	30	6.0	10.0	4.0	3.0	3.5	3.0
	2	1257	707			10.0	19.0	3.0	2.5	4.0	2.0
	3	524	377			20.0	8.0	2.5	3.0	4.5	3.5

At stage 1:

Direct Labour Cost and Overhead α_1 = Rs. 15/min.

Total Available Processing Time d_1 = 3000 mins.

At stage 2:

Direct Labour Cost and Overhead α_2 = Rs. 15/min.

Total Available Processing Time d_2 = 2500 mins.

Table 3.4: Optimal Solution of Numerical Example 6.

Index of Accepted		Quantity to be produced pcs	Machining speed m/min	
Group	Job		At Stage 1	At Stage 2
1	1	8	143.106	156.739
	2	50	133.109	115.709
2	1	100	127.29	179.907
	3	40	110.596	131.009
4	1	40	163.117	157.462
	2	50	138.144	171.259

Total Profit = Rs. 3,390.00

Total Production Cost = Rs.33,879.48

Total Processing Time Utilized:

At stage 1 = 3,000.00 mins.

At stage 2 = 2,500.00 mins.

3.5 HEURISTIC SOLUTION PROCEDURES:

In Sec. 3.4, solution methodologies for two variations of the problem, namely, prespecified unit processing time and machining speeds are decisions variables, were developed. The objective was to maximize the total profit. These solution methodologies used the basic concept of B & B for providing the optimal solution when at least on one of the production stages, the total processing time required to process all the jobs to their lot sizes was greater than the available processing time on that stage.

It was realized on solving many multistage production system problems using the solution procedures given in Sec.3.4, that the computational effort increases with the number of the production stages and the number of jobs available for production. Therefore, an effort was made to reduce the computational burden involved in the use of B & B procedure. This was achieved by reducing the size of the problem to be handled by B & B procedure. The reduction in the size of the problem is achieved by identifying a set of jobs which seem to have high potential for appearing in the optimal solution generated by the solution methodology developed in Sec. 3.4.1.3. The heuristic procedure is described as follows.

As soon as, the first step of the algorithms, described in Sec. 3.4, indicates that there is at least one production stage on which the available processing time is less than the total time required to process all the jobs to their lot sizes, contribution of each job towards the objective function, i.e., maximization of total profit is calculated. A job with the highest contribution is identified and is selected for inclusion in the list of selected jobs provided the available processing time is sufficient to process this job to its lot size. After the first job is selected, the next job with highest contribution is identified and the procedure is repeated till no job can be selected for inclusion in the list of selected jobs.

If the number of selected jobs in this list is less than three, the list is made empty and the B & B procedure, i.e., Step 3 onwards of the algorithm given in Sec. 3.4, are followed as such. However, if the list of selected jobs contains more than two jobs, the last two jobs are removed from the list. The Step 3 of B & B procedure is suitably modified for each of the algorithms given in Sec. 3.4 to incorporate the heuristic procedure into the B & B procedure. The remaining steps of the algorithms, i.e., Step 4 onwards remain the same. The following section gives the modified Step 3 for the various cases.

3.5.1 Total Profit Maximization When Unit Processing Time is Prespecified:

3.5.1.1 Heuristic Algorithm:

The Step 3 of the original algorithm given in Sec. 3.4.1.4 is modified as follows:

Step 3.1: Select an initial node: $n = 1$, $N1 = \{n\}$. For this node, set $R = U$, $S = \emptyset$, $s_k = d_k$ and $D_k = 0 \ \forall \ k$, $\underline{Z} = 0$ and $X_i = 1 \ \forall \ i$. Go to Step 3.2.

Step 3.2: Calculate,

$$N_{ij} = \min_k \, [s_k / \{(S_{ik} \, X_i + s_{ijk})/l_{ij} + p_{ijk}\}] \ \forall \ J_{ij} \varepsilon \ R$$

Find $\max\limits_{J_{ij} \varepsilon R} \; B_{ij} \, N_{ij}$ and denote the corresponding job by $J_{\bar{i}\bar{j}}$. For each production stage, calculate

$\bar{D}_k = D_k + S_{\bar{i}k} \, X_i + s_{\bar{i}\bar{j}k} + w_{\bar{i}\bar{j}} \, p_{\bar{i}\bar{j}k}$, where $w_{\bar{i}\bar{j}} = \max w_{ij}$; w_{ij} is varied from zero to $l_{\bar{i}\bar{j}}$ till $\bar{D}_k \leq d_k \ \forall \ k$.

Go to Step 3.3.

<u>Step 3.3</u>: If $w_{\bar{i}\bar{j}} = l_{\bar{i}\bar{j}}$ and $\bar{D}_k = d_k \; \forall \; k$, set $S^* = S + \{J_{\bar{i}\bar{j}}\}$,
$\underline{Z} = \sum_{J_{ij} \varepsilon S^*} B_{ij} \; l_{ij}$ and go to Step 8. If $w_{\bar{i}\bar{j}} = 0$ or
$w_{\bar{i}\bar{j}} < l_{\bar{i}\bar{j}}$, go to Step 3.4; otherwise,
set $R = R - \{J_{\bar{i}\bar{j}}\}$, $S = S + \{J_{\bar{i}\bar{j}}\}$, $D_k = \bar{D}_k, s_k = d_k - D_k \; \forall \; k$
and $X_{\bar{i}} = 0$.
Go to Step 3.2.

<u>Step 3.4</u>: If $S = \emptyset$ and $w_{\bar{i}\bar{j}} = 0$; set $Z^* = 0$, $S^* = \emptyset$ and stop.
If set S contains more than two jobs, remove the last two jobs selected from the list S and include them in R; otherwise, make $S = \emptyset$. Set $X_i = 1$ for those groups G_i, of which no job is included in S.
Set $\underline{Z} = \sum_{J_{ij} \varepsilon S} B_{ij} \; l_{ij}$, $D_k = \{ \sum_{J_{ij} \varepsilon S} (s_{ijk} + w_{ij} \; p_{ijk})$
$+ \sum_{i=1}^{N_g} S_{ik} \; X_i \}$ and $s_k = d_k - D_k \; \forall \; k$. Recalculate the value of $N_{ij} \; \forall \; J_{ij} \; \varepsilon \; R$. Find $\max_{J_{ij} \varepsilon R} B_{ij} \; N_{ij}$ and denote the corresponding job by $J_{\bar{i}\bar{j}}$. Set $\bar{Z} = \underline{Z} + B_{\bar{i}\bar{j}} \; N_{\bar{i}\bar{j}}$ and go to Step 4.

3.5.2 <u>Total Profit Maximization when Machining Speeds are Decision Variables</u>:

3.5.2.1 <u>Heuristic algorithm</u>:

Step 3 remains the same as given in Sec. 3.5.1.1 except the fact that instead of p_{ijk}, $p_{ijk}^{(t)}$ is used for calculating N_{ij}. Note that $p_{ijk}^{(t)}$ represents the maximum-production-rate unit processing time.

3.5.3 Illustrative Example and Computational Experience:

3.5.3.1 Example:

Numerical example 5 presented in Sec. 3.4.1.5 is considered to demonstrate the heuristic solution procedure.

3.5.3.2 Solution:

The following is the stepwise solution of the example using the heuristic approach.

Step 1: Q_1 = 3105.00 mins, and Q_2 = 2995.00 mins.

Since $Q_k > d_k \; \forall \; k$, go to Step 3.

Step 3.1: Select an initial node: set $n = 1$, $N1 = \{n\} = \{1\}$.

For node 1, set $R = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$, $S = \phi$, $D_k = 0 \; \forall \; k$, s_k= 2400.00 min $\forall$ k, $\underline{Z} = 0$ and $X_i = 1 \; \forall \; i$.

Go to Step 3.2.

Step 3.2: Calculate $N_{ij} \; \forall \; J_{ij} \in R$. Find $\max\limits_{J_{ij} \in R} B_{ij} N_{ij}$. Its value is Rs. 4080.00 corresponding to job J_{12}.

We find that $\overline{D}_1$ = 68.00 mins and $\overline{D}_2$ = 94.00 mins for w_{12} = 10 pcs.

Go to Step 3.3.

Step 3.3: Since $\overline{D}_k < d_k \; \forall \; k$ and $w_{12} = l_{12}$, set $R = R-\{J_{12}\}$ $= \{J_{11}, J_{21}, J_{22}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$,

$S = S + \{J_{12}\} = \{J_{12}\}$, $X_1 = 0, D_1$ = 68.00 mins,

D_2 = 94.00 mins, s_1 = 2332.00 mins. and s_2 = 2306.00 mins.

Go to Step 3.2.

Now we recalculate the value of N_{ij} $\forall$ $J_{ij} \varepsilon R$. It is observed that $\max\limits_{J_{ij}\varepsilon R} B_{ij} N_{ij}$ occurs corresponding to a job J_{22}. We find that $\bar{\bar{D}}_1 = 562.00$ mins, $\bar{\bar{D}}_2 = 473.00$ mins and $w_{22} = 50$ pcs. Go to Step 3.3.

Since $w_{22} = l_{22}$ and $\bar{\bar{D}}_k < d_k \ \forall \ k$, set

$R = \{J_{11}, J_{21}, J_{23}, J_{31}, J_{32}, J_{41}, J_{42}, J_{43}\}$,

$S = \{J_{12}, J_{22}\}$, $X_2 = 0$, $D_1 = 562.00$ mins, $D_2 = 473.00$ mins, $s_1 = 1838.00$ mins. and $s_2 = 1927.00$ mins.

The Step 3.2 and 3.3 are followed iteratively and sets R & S are generated.

$R = \{J_{31}, J_{41}, J_{42}\}$ and

$S = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{23}, J_{32}, J_{43}\}$. Further,

$D_1 = 2394.00$ mins, $D_2 = 2386.00$ mins, $X_i = 0 \ \forall$ i and

$w_{42} = 0$.

Since $w_{42} = 0$ and $D_k < d_k$, we go to Step 3.4.

Step 3.4: We remove the last two jobs J_{23} and J_{43} from S and include them in R.

Now, set $R = \{J_{23}, J_{31}, J_{41}, J_{42}, J_{43}\}$,

$S = \{J_{11}, J_{12}, J_{21}, J_{22}, J_{32}\}$, $X_4 = 1$,

$\underline{Z}$ = Rs. 2210.00, $D_1 = 1504.00$ mins,

$D_2 = 1536.00$ mins, $s_1 = 896.00$ mins and

$s_2 = 864.00$ mins.

Recalculate N_{ij} $\forall$ J_{ij} ε R. Find $\max_{J_{ij} \varepsilon R} B_{ij} N_{ij}$. Its value is Rs. 828.00 corresponding to job J_{23}.

Set $\bar{\bar{Z}} = \underline{\underline{Z}} + B_{23} N_{23}$ = Rs. 3038.00 and go to Step 4.

Now we follow B & B procedure from Step 4 onwards as discussed in Sec. 3.4.1.4. The final product mix obtained is as shown in Table 12.

3.5.3.3 Computational experience:

It was observed that the solution of the numerical example 5 obtained by heuristic and optimizing procedures turns out to be the same. The heuristic and optimizing solution procedures required a CPU time of 0.20 secs. and 0.26 secs. respectively for solving this example on DEC-1090 computer system. Number of problems were solved using both the approaches. In each case, it was observed that the heuristic approach requires less computational effort as compared to the optimizing procedure.

3.6 VARIATIONS OF MULTI-STAGE PROBLEM:

In Sec. 3.4 and 3.5, we presented mathematical models and solution procedures for the multi-stage production systems and the objective was to maximize the total profit. The mathematical models and solution procedures can be suitably modified for maximization of total number of units produced. The term B_{ij}, i.e., the per unit profit for job J_{ij}, is replaced by 1. When Ns is put equal to 1, the multi-stage problem becomes single stage problem.

The multi-stage algorithms given in Secs. 3.4 and 3.5 were used to solve the single stage numerical examples 1,2,3 and 4 and it was observed that the solutions obtained were same as obtained by algorithms given in Sec. 2.4.

The solution procedure given in Sec. 3.4.2 is based on the fact that the unit processing times of the jobs are determined by machining speeds which are treated as decision variables. However, there may be multi-stage problems in which the unit processing time for some jobs are prespecified while for the remaining jobs the unit processing time depend upon the machining speeds which are also to be optimally selected.

Initially, we assume that the jobs for which the machining speeds are not prespecified, are processed using minimum unit production time machining speeds. The machine loading and product mix is determined using the algorithm given in Sec. 3.4.1. Using this product mix, the time required on each production stage to process the selected jobs for which the unit processing time is not prespecified, is calculated and subtracted from the available processing time for that stage. The remaining available processing time at each stage is used to obtain the optimal machining speeds for the various jobs (for which the processing times are not prespecified) to be processed on that stage.

The multi-stage problem is now converted into multiple single stage problems (the number of single stage problems considered being equal to the number of stages in the multi-stage problem). The optimal machining speeds are determined using the basic concepts given in Sec. 2.4.1.2.

CHAPTER IV

CONCLUSIONS AND SCOPE FOR FURTHER WORK

4.1 CONCLUSIONS:

In this thesis, we have developed mathematical models and solution procedures for the machine loading and product mix decision problems for single and multi-stage production systems with limited available production times at each stage. The objective was to maximize either the total profit or the total amount of production. Two important variations of the problem, namely, the unit production time is prespecified and machining speeds are decision variables, were also considered.

For each one of the above stated variation of the problem, an optimizing algorithm which utilized the basic framework of branch and bound procedure, was developed. In order to reduce the computational effort, heuristic algorithms were also developed for each of the problem variations. Sample problems were solved using both the optimizing and heuristic algorithms.

4.2 SCOPE FOR FURTHER WORK:

In order to account for realistic production situations, the single and multi-stage models developed in this thesis need to be extended to account for the following situations:

i) Each production stage has one or more machines.

ii) Machines are capable for processing more than one unit at a time.

iii) The jobs may have technological ordering for the operations to be performed on various production stages.

iv) The jobs may have due dates associated with them.

In Section 2.4.1.2, a technique was suggested for determining the step length for decreasing or increasing the value of μ. Better methodology needs to be developed for appropriate selection of step length; since this will considerably reduce the computational effort involved in the use of various algorithms.

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